Using Recursive Utility to Assess Uncertainty

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Authors: Jaroslav Borovicka (NYU), Lars Peter Hansen (University of Chicago), and Thomas J. Sargent (NYU)

> “Uncertainty is the only certainty there is, and knowing how to live with insecurity is the only security.” - John Allen Paulos

11.1. Introduction

This chapter studies the recursive utility preference specification of [Kreps and Porteus, 1978] and [Epstein and Zin, 1989]. We include interpretations of this specification proposed by [Hansen and Sargent, 2001] and [Anderson *et al.*, 2003] that are designed to capture concerns about model misspecification. We deploy two distinct approximation approaches. One approach builds on a characterization by [Duffie and Epstein, 1992] uses a continuous-time limiting approximation to a discrete-time specification in which underlying shocks are normally distributed. Our account of the approximation differs in details from that of [Duffie and Epstein, 1992]. We represent the limiting approximation with a Brownian motion information structure. Our second approximation lets macroeconomic uncertainty have first-order consequences. We modify first- and second-order approximations routinely used in the macroeconomics literature in ways designed to focus on macroeconomic uncertainty and explore implications of (nonstandard) first- and second-order approximations to equilibria of dynamic stochastic models. Our approximations apply to production-based macro-finance models in which there are opportunities to invest in different kinds of capital. Our approximation approaches use a change in probability measure to represent a decision maker’s adjustments for uncertainty. We shall see that this adjustment differs from the one underlying the so-called risk neutral distribution that is widely used to price derivative claims. Our change of measure instead emerges from specifications of preferences like those in [Hansen and Sargent, 2001] and [Anderson *et al.*, 2003] that build on a robust control literature initiated by [Jacobson, 1973] and [Whittle, 1981].

We extend work by [Schmitt-Grohé and Uribe, 2004] and [Lombardo and Uhlig, 2018] in ways that highlight consequences of uncertainty. We design approximations to make implied stochastic discount factors reside within an exponential linear quadratic class, a class that gives rise to tractable formulas for asset valuation over alternative investment horizons. See, for instance, [Ang and Piazzesi, 2003] and [Borovička and Hansen, 2014]. The class is also useful for studying production-based macro-finance models with opportunities to invest in different forms of capital.

Note

We recognize that nonlinearities can be more accurately captured by global solution methods. Sometimes these are too costly or just not feasible.

11.2. Recursive utility valuation process

We construct continuation value and stochastic discount processes, important constituents of many dynamic stochastic models in macroeconomics and finance.

11.2.1. Basic recursion

A homogeneous of degree one representation of recursive utility is

$$V_t = \left[(1-\beta)(C_t)^{1-\rho} + \beta(R_t)^{1-\rho}\right]^{\frac{1}{1-\rho}} \tag{11.1}$$

where

$$R_t = \left(\mathbb{E}\left[(V_{t+1})^{1-\gamma} \mid \mathfrak{A}_t\right]\right)^{\frac{1}{1-\gamma}}. \tag{11.2}$$

Value V_t defined in equation (11.1) is a homogeneous of degree one function of C_t and R_t; equation (11.2) defines R_t as a homogeneous of degree one function of another function of V_{t+1}.

In equation (11.1), $0 < \beta < 1$ is a subjective discount factor and $\frac{1}{\rho}$ is the elasticity of intertemporal substitution, while γ in equation (11.2) describes attitudes towards risk.

Continuation values are determined only up to an increasing transformation. For computational and conceptual reasons, it is useful to work with the transformation $\widehat{V}_t = \log V_t$. Recursions for $\widehat{V}_t$ expressed in terms of the logarithm of consumption $\widehat{C}_t$ are

$$\widehat{V}_t = \frac{1}{1-\rho} \log \left[(1-\beta) \exp[(1-\rho)\widehat{C}_t] + \beta \exp\left[(1-\rho)\widehat{R}_t \right] \right] \tag{11.3}$$

where

$$\widehat{R}_t = \frac{1}{1-\gamma} \log \mathbb{E} \left[\exp\left((1-\gamma)\widehat{V}_{t+1} \right) \mid \mathfrak{A}_t \right]. \tag{11.4}$$

The right side of recursion (11.3) is the logarithm of a constant elasticity of substitution (CES) function of $\exp(\widehat{C}_t)$ and $\exp(\widehat{R}_t)$.

Remark 11.1

The limit of $\widehat{R}_t$ as γ approaches 1 is expected logarithmic utility:

$$\lim_{\gamma \downarrow 1} \widehat{R}_t = \lim_{\gamma \downarrow 1} \frac{\log \mathbb{E}\left(\exp\left[(1-\gamma)\widehat{V}_{t+1} \right] | \mathfrak{A}_t \right)}{1-\gamma} = \mathbb{E}\left(\widehat{V}_{t+1} | \mathfrak{A}_t \right).$$

We shall construct small noise expansions of $\widehat{V}_t$ and $\widehat{R}_t$ separately, then assemble them. Before doing so, we offer a reinterpretation of our recursions (11.3)-(11.4).

11.2.2. Preference for robustness

When $\gamma > 1$, (11.4) emerges as an indirect utility function for a robust control problem in which $\frac{1}{\gamma-1}$ serves as a penalty parameter on entropy of a baseline model relative to an alternative model.

The interpretation of (11.4) as an indirect utilty function from a minimization problem originated in [Hansen and Sargent, 1995], which rested on earlier work by [Jacobson, 1973] and [Whittle, 1981].

Let the random variable $N_{t+1} \geq 0$ satisfy $\mathbb{E}\left(N_{t+1} \mid \mathfrak{A}_t \right) = 1$ so that it is a likelihood ratio. Think of replacing the expected continuation value $\mathbb{E}\left(\widehat{V}_{t+1} \mid \mathfrak{A}_t \right)$ by

$$\min_{N_{t+1} \geq 0, \mathbb{E}(N_{t+1} | \mathfrak{A}_t) = 1} \mathbb{E}\left(N_{t+1} \widehat{V}_{t+1} \mid \mathfrak{A}_t \right) + \xi \mathbb{E}\left(N_{t+1} \log N_{t+1} \mid \mathfrak{A}_t \right). \tag{11.5}$$

where ξ is a parameter that penalizes departures of N_{t+1} from unity as measured by relative entropy. Conditional entropy relative to an alternative conditional probability induced by applying a change of measure N_{t+1} is

$$\mathbb{E}\left(N_{t+1}\log N_{t+1}\mid \mathfrak{A}_t\right) \geq 0,$$

which, because $n\log n$ is a convex function, follows from Jensen's inequality. Evidently, this inequality becomes an equality when $N_{t+1}=1$.

Relative entropy serves as a summary measure of the difference between two probability distributions. We can think of N_{t+1} as a relative conditional likelihood ratio of an alternative model vis-a-vis a baseline model. Then $\mathbb{E}\left(N_{t+1}\log N_{t+1}\mid \mathfrak{A}_t\right)$ is an expected (conditional) log-likelihood ratio of the alternative model when the expectation is taken using the alternative probability model. A small expected log likelihood indicates a small discrepancy between two models, i.e., two probability distributions.

Remark 11.2

To solve minimization problem (11.5), attach a Lagrange multiplier ℓ to the constraint and form a Lagrangian. Then minimize with respect to the random variable N and maximize it with respect to ℓ. This extremization problem separates across states, so to minimize with respect to N we can solve the following minimum problem for each possible n

$$\min_n n\hat{v} + \xi n\log n + \ell(n-1).$$

Here n is a potential realization of N_{t+1} and v is a potential realization of $\widehat{V}_{t+1}$. First-order necessary conditions are:

$$\hat{v} + \xi + \xi\log n + \ell = 0,$$

which implies the minimizer

$$n^* = \exp\left[-\frac{1}{\xi}\left(\hat{v}+\ell+\xi\right)\right],$$

and the minimized objective

$$-\xi\exp\left[-\frac{1}{\xi}\left(\hat{v}+\ell+\xi\right)\right] - \ell.$$

To determine ℓ, we pose

$$\max_\ell -\xi\mathbb{E}\left(\exp\left[-\frac{1}{\xi}\left(\widehat{V}_{t+1}+\ell+\xi\right)\right]\mid \mathfrak{A}_t\right) - \ell$$

whose first-order necessary condition is

$$\mathbb{E}\left(\exp\left[-\left(\frac{1}{\xi}\right)\widehat{V}_{t+1}\right] \mid \mathfrak{A}_t\right)\exp\left[-\left(\frac{\ell+\xi}{\xi}\right)\right] - 1 = 0.$$

The maximizing ℓ is

$$\ell^* = \xi \log \mathbb{E}\left(\exp\left[-\left(\frac{1}{\xi}\right)\widehat{V}_{t+1}\right] \mid \mathfrak{A}_t\right) - \xi$$

and the minimized objective function is

$$-\xi \log \mathbb{E}\left(\exp\left[-\left(\frac{1}{\xi}\right)\widehat{V}_{t+1}\right] \mid \mathfrak{A}_t\right).$$

The minimizing N_{t+1} is

$$N^*_{t+1} = \frac{\exp\left(-\frac{1}{\xi}\widehat{V}_{t+1}\right)}{\mathbb{E}\left[\exp\left(-\frac{1}{\xi}\widehat{V}_{t+1}\right) \mid \mathfrak{A}_t\right]}. \tag{11.6}$$

The minimizer (11.6) of problem (11.5) evidently shifts probabilities toward low continuation values, so that probabilities of undesirable events are raised and probabilities of desirable events are lowered. [Bucklew, 2004] called this a stochastic version of Murphy's law. Notice that the minimized objective satisfies

$$-\xi \log \mathbb{E}\left[\exp\left(-\frac{1}{\xi}\widehat{V}_{t+1}\right) \mid \mathfrak{A}_t\right] = \widehat{R}_t$$

where earlier we described $\widehat{R}_t$ by equation (11.4) after we set $\xi = \frac{1}{\gamma-1}$.
It follows from (11.6) that

$$N^*_{t+1} = \exp\left[-\frac{1}{\xi}\left(\widehat{V}_{t+1} - \widehat{R}_t\right)\right].$$

The random variable N^*_{t+1} will appear often below.

11.2.3. Stochastic discount factor process

A stochastic discount factor (SDF) process $S = \{S_t : t \geq 0\}$ describes a consumer's attitudes about small changes in uncertainty. SDF processes have several uses. First, they contain shadow prices that tell how a consumer's attitudes about uncertainty shape marginal valuations of risky assets. Second, they shape first-order necessary conditions for optimally choosing financial and physical investments. Third, they underlie tractable formulas for equilibrium asset prices. Fourth, they can help construct Pigouvian taxes that ameliorate externalities under uncertainty. Fifth, they can help evaluate effects of small (local) changes in government policies.

To deduce an SDF process, we posit that a date zero value of a risky date t consumption payout χ_t is

$$\pi_0^t(\chi_t) = E\left[\left(\frac{S_t}{S_0}\right)\chi_t \Big| \mathfrak{A}_0\right]. \tag{11.7}$$

To compute the ratio $\frac{S_t}{S_0}$ that appears in formula (11.7) we evaluate the slope of an indifference curve that connects a baseline consumption process $\{C_t\}_{t=0}^{\infty}$ to a perturbed consumption process

$$(C_0 - P_0(\mathsf{q}), C_1, C_2, \ldots, C_t + \mathsf{q}\chi_t, C_{t+1}, \ldots).$$

We think of q as parameterizing an indifference curve, so $P_0(\mathsf{q})$ expresses how much current period consumption must be reduced to keep a consumer on the same indifference curve after we replace C_t by $C_t + \mathsf{q}\chi_t$. We set $\pi_0^t(\chi_t)$ defined in equation (11.7) equal to the slope of that indifference curve:

$$\pi_0^t(\chi_t) = \frac{d}{d\mathsf{q}} P_0(\mathsf{q})\Big|_{\mathsf{q}=0}.$$

For recursive utility, a one-period increment in the stochastic discount factor process is

$$\begin{aligned}\frac{S_{t+1}}{S_t} =& \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \exp\left[(1-\gamma)\left(\widehat{V}_{t+1} - \widehat{R}_t\right)\right] \exp\left[(\rho-1)\left(\widehat{V}_{t+1} - \widehat{R}_t\right)\right] \\ =& \beta N_{t+1}^* \exp\left(\widehat{S}_{t+1} - \widehat{S}_t\right),\end{aligned} \tag{11.8}$$

where

$$\widehat{S}_{t+1} - \widehat{S}_t \stackrel{\text{def}}{=} -\rho\left(\widehat{C}_{t+1} - \widehat{C}_t\right) + (\rho - 1)\left(\widehat{V}_{t+1} - \widehat{R}_t\right) \tag{11.9}$$

and N_{t+1}^* induces the change of probability measure that equation (11.6) presents as the outcome of a robust valuation problem. By design, the construction of $\widehat{S}_{t+1} - \widehat{S}_t$ captures the terms involving ρ. We will use the second line in (11.8) in what follows. We will think of β as a subjective discount factor adjustment, N_{t+1}^* as a change-of-measure adjustment for uncertainty, and $\exp\left(\widehat{S}_{t+1} - \widehat{S}_t\right)$ as an adjustment for the elasticity of intertemporal substitution. We interpret the twisted transition probability measure induced by N_{t+1}^* as an adjustment for uncertainty about evaluations.

The recursive structure of preferences makes the time-t stochastic discount factor $\frac{S_t}{S_0}$ the product of the respective one-period stochastic discount factor increments. Similarly, we can compound one-period transition uncertainty measures into multiple time-horizon measures of uncertainty.

Remark 11.3

To verify formula (11.8), we compute a one-period intertemporal marginal rate of substitution. Given the valuation recursions (11.3) and (11.4), we construct two marginal utilities, one each for CES and exponential one-period utility functions:

$$mc = (1-\beta)(c)^{-\rho} \exp\left[(\rho - 1)\hat{v}\right]$$

$$m\hat{r} = \beta \exp[(1-\rho)\left(\hat{r} - \hat{v}\right)]$$

From the certainty equivalent formula, we construct the marginal utility of the next-period logarithm of the continuation value:

$$m\hat{v}^{+} = \exp\left[(1-\gamma)\left(\hat{v}^{+} - \hat{r}\right)\right]$$

where the $+$ superscript is used to denote the next-period counterpart. The next-period marginal utility of consumption is

$$mc^{+} = (1-\beta)\left(c^{+}\right)^{-\rho} \exp\left[(\rho - 1)\hat{v}^{+}\right]$$

Putting these four formulas together using the chain rule for differentiation gives a marginal rate of substitution:

$$\frac{(m\hat{r}^{+})(m\hat{v}^{+})(mc^{+})}{mc} = \beta\left(\frac{c^{+}}{c}\right)^{-\rho} \exp\left[(1-\gamma)\left(\hat{v}^{+} - \hat{r}\right)\right] \exp\left[(\rho-1)\left(\hat{v}^{+} - \hat{r}\right)\right].$$

Now let $\hat{v}^{+} = \widehat{V}_{t+1}$, $c^{+} = C_{t+1}$, $C_t = c$ and $\hat{r} = \widehat{R}_t$ to obtain the formula for the one-period stochastic discount factor (11.8).

11.3. Continuous-time limit

We now study a continuous-time limit that approximates a discrete-time specification. Since we continue to work with normal shocks, the continuous-time counterparts are Brownian increments. The continuation value in continuous time will evolve as:

$$dV_t = V_t \mu_t^V dt + V_t \sigma_t^V \cdot dW_t$$

for some drift (local mean), $V_t \mu_t^V$ and some local shock exposure vector $V_t \sigma_t^V$, where $\{W_t : t \geq 0\}$ is a multivariate Brownian motion. Scaling the local evolution coefficients by V_t is convenient when the continuation value process is presumed to be positive. As in discrete time, it is convenient to work with the logarithm of the continuation value process (the log is a strictly increasing transformation). The implied evolution is

$$d\widehat{V}_t = \hat{\mu}_t^V dt + \sigma_t^V \cdot dW_t$$

where $\hat{\mu}_t = \mu_t^V - \frac{1}{2} \mid \sigma_t^V \mid^2$. This adjustment follows from Ito's formula.

11.3.1. Discrete-time approximation

To study the utility recursion, start with a discrete-time specification:

$$\begin{aligned}
&\frac{1}{1-\rho}\log\left[(1-\beta_\epsilon)\exp\left[(1-\rho)\left(\widehat{C}_t - \widehat{V}_t\right)\right] + \beta_\epsilon \exp\left[(1-\rho)\left(\widehat{R}_t - \widehat{V}_t\right)\right]\right] = 0 \\
&\widehat{R}_t - \widehat{V}_t = \frac{1}{1-\gamma}\log\mathbb{E}\left(\exp\left[(1-\gamma)\left(\widehat{V}_{t+\epsilon} - \widehat{V}_t\right)\right] \mid \mathfrak{A}_t\right)
\end{aligned}$$

where $\beta_\epsilon = \exp(-\delta\epsilon)$ and $\delta > 0$ is the instantaneous subjective rate of discount. Consider the time derivative of the second recursion:

$$\begin{aligned}
\frac{d}{d\epsilon}\left(\widehat{R}_t - \widehat{V}_t\right)\Big|_{\epsilon=0} &= \frac{d}{d\epsilon}\frac{1}{1-\gamma}\log\mathbb{E}\left(\exp\left[(1-\gamma)\left(\widehat{V}_{t+\epsilon} - \widehat{V}_t\right)\right] \mid \mathfrak{A}_t\right)\Big|_{\epsilon=0} \\
&= \hat{\mu}_t^V + \frac{(1-\gamma)}{2}|\sigma_t^V|^2 \\
&= \mu_t^V - \frac{\gamma}{2}|\sigma_t^V|^2.
\end{aligned}$$

by local log normality.

We turn now to the first recursion and compute time derivatives in three steps. First, we evaluate the term inside the logarithm as ϵ tends to zero :

$$\lim_{\epsilon \downarrow 0} \exp\left[(1-\rho)\left(\widehat{R}_t - \widehat{V}_t\right)\right] = 1.$$

This term is in the denominator as implied by the derivative with respect to a logarithm. Second, we differentiate the term inside the logarithm with respect to ϵ as contributed by β_ϵ:

$$\delta \exp\left[(1-\rho)\left(\widehat{C}_t - \widehat{V}_t\right)\right] - \delta.$$

Third, we differentiate the term inside the logarithm with respect to ϵ as contributed by

$$\frac{d}{d\epsilon}\exp\left[(1-\rho)\left(\widehat{R}_t-\widehat{V}_t\right)\right]=\hat{\mu}_t^V+\frac{(1-\gamma)}{2}|\sigma_t^V|^2.$$

Putting together the derivative components gives:

$$\begin{aligned} 0 &= \frac{\delta\left[\left(\frac{C_t}{V_t}\right)^{1-\rho}-1\right]}{1-\rho}+\hat{\mu}_t^V+\frac{(1-\gamma)}{2}|\sigma_t^V|^2 \\ &= \frac{\delta\left[\left(\frac{C_t}{V_t}\right)^{1-\rho}-1\right]}{1-\rho}+\mu_t^V-\frac{\gamma}{2}|\sigma_t^V|^2. \end{aligned} \tag{11.10}$$

This relation imposes a restriction across the local mean, μ_t^V and local variance, $|\sigma_t^V|^2$ of the continuation value. [Duffie and Epstein, 1992] refer to γ as a *variance multiplier* where larger values of γ imply a more substantial adjustment for local volatility. As in discrete-time, the $\rho=1$ case is interesting special case represented as;

$$0=-\delta\left(\widehat{V}_t-\widehat{C}_t\right)+\hat{\mu}_t^V+\frac{(1-\gamma)}{2}|\sigma_t^V|^2.$$

11.3.2. Robustness to misspecification

To investigate an aversion to model misspecification in continuous time, we now treat the distribution of $\{W_t : t\geq 0\}$ as uncertain. We allow for probability measures that entertain possible Brownian motions with local means or drifts that are history dependent.

We start by considering positive martingales $\{M_t^H : t\geq 0\}$ parameterized by alternative $\{H_t : t\geq 0\}$ processes with the same dimension as the underlying Brownian motion.
The martingales have local evolutions:

$$dM_t^H=M_t^H H_t\cdot dW_t,$$

and we initialize them at $M_0^H=1$. Observe that by applying Ito's formula, $\log M^H$ evolves as:

$$d\log M_t^H=-\frac{1}{2}|H_t|^2dt+H_t\cdot dW_t.$$

We use these martingales as relative densities or likelihood ratios.

Write the discrete-time counterpart as

$$\log M^H_{t+\epsilon} - \log M^H_t = -\frac{\epsilon}{2}|H_t|^2 + H_t \cdot (W_{t+\epsilon} - W_t)$$

Let w be a realized $W_{t+\epsilon} - W_t$ and h be a realization H_t. Then $\log M_{t+\epsilon} - \log M_t$ contributes $-\frac{\epsilon}{2}h'h + h \cdot w$ to the log-likelihood. The standard normal density for $\sqrt{\epsilon}\,(W_{t+\epsilon} - W_t)$ contributes $-\frac{1}{\epsilon}w'w - \log(2\pi\epsilon)$. Put these two components together, we have a log-likelihood:

$$-\frac{\epsilon}{2}h'h + h \cdot w - \frac{1}{2\epsilon}w'w - \log(2\pi\epsilon) = -\frac{1}{2\epsilon}(w - \epsilon h)'(w - \epsilon h) - \log(2\pi\epsilon).$$

The altered conditional density has mean ϵh, which is the realized value of ϵH_t with the same conditional covariance matrix as before. Moreover, the conditional expectation of

$$\mathbb{E}\left[\left(\frac{M^H_{t+\epsilon}}{M^H_t}\right)\left(\log M^H_{t+\epsilon} - \log M^H_t\right) \mid \mathfrak{A}_t\right] = \frac{\epsilon}{2}|H_t|^2,$$

which measures the statistical divergence or relative entropy between original and altered conditional probabilities.

For the continuous-time limit, under the H change of probability measure:

$$dW_t = H_t dt + d\widetilde{W}^H_t$$

where $\widetilde{W}^H$ is a standard Brownian motion. Thus, potential changes of probability measures induce local means or drift H processes to the Brownian motion. The continuous-time counterpart to conditional relative entropy at time t is $\frac{1}{2}|H_t|^2$.

We can justify focusing on drift distortions for Brownian increments because of our imposition of absolute continuity of the alternative probabilities with respect to the baseline specification of a multivariate standard Brownian motion. This is an implication of the Girsanov Theorem.

We can now deduce a robustness adjustment in continuous time. Consider formula (11.10) when $\gamma = 1$ modified for a potential change in the probability measure

$$0 = \frac{\delta\left[\left(\frac{C_t}{V_t}\right)^{1-\rho} - 1\right]}{1-\rho} + \hat{\mu}^V_t + \sigma^V_t \cdot H_t$$

Modify this equation to include minimization over H_t subject to a relative entropy penalty $\frac{\xi}{2}|H_t|^2$:

$$0 = \min_{H_t} \frac{\delta\left[\left(\frac{C_t}{V_t}\right)^{1-\rho} - 1\right]}{1-\rho} + \hat{\mu}^V_t + \sigma^V_t \cdot H_t + \frac{\xi}{2}H_t \cdot H_t.$$

The minimizer is

$$H_t^* = -\frac{1}{\xi}\sigma_t^V$$

with a minimized objective

$$0 = \frac{\delta\left[\left(\frac{C_t}{V_t}\right)^{1-\rho} - 1\right]}{1-\rho} + \hat{\mu}_t^V - \frac{1}{2\xi}\left|\sigma_t^V\right|^2$$

Notice that this agrees with formula (11.10) for $\gamma - 1 = 1/\xi$. The explicit link is entirely consistent with our discrete-time equivalence result. By taking the continuous-time limit, we are able to focus our misspecification analysis on changing local means of the underlying Brownian increments.

11.3.3. Uncertainty pricing

To prepare the way for studying valuations, we compound the equilibrium version of $\{H_t^* : t \geq 0\}$ to get an exponential martingale:

$$M_t^* = \exp\left(\int_0^t H_\tau^* dW_\tau - \frac{1}{2}\int_0^t \left|H_\tau^*\right|^2 d\tau\right)$$

provided that the constructed process is a martingale.[1] With this construction we interpret $-H_t^*$ as the vector of local uncertainty prices that give compensations for exposure to Brownian increment uncertainty. These compensations are expressed as changes in conditional means under the baseline distribution, as is typical in continuous-time asset pricing.

In a model comparison paper, [Hansen *et al.*, 2024] use a continuous-time specification as we described here and discuss the continuous time methods for shock elasticities as barometers for the alternative models. The models include ones with two capital stocks differentially exposed to uncertainty and ones with two types of agents differentially exposed to financing constraints.

11.4. Small noise expansion of dynamic stochastic equilibria

We now study another characterization that sometimes helps provide good approximations of equilibria of dynamic stochastic models. While the approximations are based on derivations in [Schmitt-Grohé and Uribe, 2004] and [Lombardo and Uhlig, 2018], we extend them in ways that emphasize uncertainty impacts that are reflected even in first-order contributions. By design, our approximations of stochastic discount factors reside within the exponential linear quadratic class, a class known to provide tractable formulas for asset valuations across investment horizons. See, for instance, [Ang and Piazzesi, 2003] and [Borovička and Hansen, 2014]. Furthermore, these approximations apply to production-based macro-finance models with investment opportunities in alternative types of capital.

While these approximations are tractable, we recognize that they might omit or disguise important aspects of uncertainty. Without question, global solution methods capture important nonlinearities more accurately for some models. Nevertheless, the approximations to be presented here shed light on the structures of preferences and their implications for asset pricing in both endowment and production economies.

11.4.1. Approximate state dynamics

We follow [Lombardo and Uhlig, 2018] by considering the following class of stochastic processes indexed by a scalar perturbation parameter q:[2]

$$X_{t+1}(\mathsf{q}) = \psi\left[X_t(\mathsf{q}), \mathsf{q}W_{t+1}, \mathsf{q}\right]. \tag{11.11}$$

Here X is an n-dimensional stochastic process and $\{W_{t+1}\}$ is an i.i.d.~normally distributed random vector with conditional mean vector 0 and conditional covariance matrix I. We parameterize this family so that $\mathsf{q} = 1$ gives the model of interest.

We denote the zero-order expansion $\mathsf{q} = 0$ limit as:

$$X^0_{t+1} = \psi\left(X^0_t, 0, 0\right), \tag{11.12}$$

and assume that there exists a second-order expansion of X_t around $\mathsf{q} = 0$:

$$X_t(\mathsf{q}) \approx X^0_t + \mathsf{q}X^1_t + \frac{\mathsf{q}^2}{2}X^2_t \tag{11.13}$$

where X^1_t is a first-order contribution and X^2_t is a second-order contribution. In other words, the stochastic processes X^j, $j = 0, 1, 2$ are appropriate derivatives of X with respect to the perturbation parameter q evaluated at $\mathsf{q} = 0$.

In the remainder of this chapter, we shall construct instances of the second-order expansion (11.13) in which the generic random variable X_t is replaced, for example, by the logarithm of consumption, a value function, and so on.

Processes $X^j_t, j = 0, 1, 2$ have a recursive structure: first compute the stochastic process X^0_t, then the process X^1_t (it depends on X^0_t), and finally the process X^2_t (it depends on both X^0_t and X^1_t).

In this chapter, we use a prime $'$ to denote a transpose of a matrix or vector. When we include x' in a partial derivative of a scalar function it means that the partial derivative is a row vector. Consistent with this convention, let $\psi^i_{x'}$, the i^{th} entry of $\psi_{x'}$, denote the row vector of first derivatives with respect to the vector x, and similarly for $\psi^i_{w'}$. Since q is scalar, ψ^i_{q} is the scalar derivative with respect to q. Derivatives are evaluated at X^0_t, which in many examples is invariant over time, unless otherwise stated. This invariance follows when we impose a steady state on the deterministic system.

The first-derivative process obeys a recursion

$$X^1_{t+1} = \begin{bmatrix} \psi^1_{x'} \\ \psi^2_{x'} \\ \vdots \\ \psi^n_{x'} \end{bmatrix} X^1_t + \begin{bmatrix} \psi^1_{w'} \\ \psi^2_{w'} \\ \vdots \\ \psi^n_{w'} \end{bmatrix} W_{t+1} + \begin{bmatrix} \psi^1_{\mathsf{q}} \\ \psi^2_{\mathsf{q}} \\ \vdots \\ \psi^n_{\mathsf{q}} \end{bmatrix} \tag{11.14}$$

that we can write compactly as the following *first-order vector autoregression*:

$$X^1_{t+1} = \psi_{x'} X^1_t + \psi_{w'} W_{t+1} + \psi_{\mathsf{q}}$$

We assume that the matrix $\psi_{x'}$ is stable in the sense that all of its eigenvalues are strictly less than one in modulus.

It is natural for us to denote second derivative processes with double subscripts. For instance, for the double script used in conjunction with the second derivative matrix of ψ^i, the first subscript without a prime ($'$) reports the row location; the second subscript with a prime ($'$) reports the column location. Differentiating recursion (11.14) gives:

$$\begin{aligned} X^2_{t+1} = \; & \psi_{x'} X^2_t + \begin{bmatrix} {X^1_t}' \psi^1_{xx'} X^1_t \\ {X^1_t}' \psi^2_{xx'} X^1_t \\ \vdots \\ {X^1_t}' \psi^n_{xx'} X^1_t \end{bmatrix} + 2 \begin{bmatrix} {X^1_t}' \psi^1_{xw'} W_{t+1} \\ {X^1_t}' \psi^2_{xw'} W_{t+1} \\ \vdots \\ {X^1_t}' \psi^n_{xw'} W_{t+1} \end{bmatrix} + \begin{bmatrix} {W_{t+1}}' \psi^1_{ww'} W_{t+1} \\ {W_{t+1}}' \psi^2_{ww'} W_{t+1} \\ \vdots \\ {W_{t+1}}' \psi^n_{ww'} W_{t+1} \end{bmatrix} \\ & + 2 \begin{bmatrix} \psi^1_{\mathsf{q}x'} X^1_t \\ \psi^2_{\mathsf{q}x'} X^1_t \\ \vdots \\ \psi^n_{\mathsf{q}x'} X^1_t \end{bmatrix} + 2 \begin{bmatrix} \psi^1_{\mathsf{q}w'} W_{t+1} \\ \psi^2_{\mathsf{q}w'} W_{t+1} \\ \vdots \\ \psi^n_{\mathsf{q}w'} W_{t+1} \end{bmatrix} + \begin{bmatrix} \psi^1_{\mathsf{qq}} \\ \psi^2_{\mathsf{qq}} \\ \vdots \\ \psi^n_{\mathsf{qq}} \end{bmatrix}. \end{aligned} \tag{11.15}$$

Recursions (11.14) and (11.15) have a linear structure with some notable properties. The law of motion for X^0 is deterministic and is time invariant if (11.11) comes from a stationary $\{X_t\}$ process. Dynamics of X^2 are nonlinear only in X^1 and W_{t+1}. Thus, stable dynamics for X^1 that prevail when $\psi_{x'}$ is a stable matrix imply stable dynamics for X^2.

Remark 11.4

Perturbation methods have been applied to many rational expectations models in which partial derivatives of ψ with respect to q are often zero.[3] However, derivatives of ψ with respect to q are not zero in production-based equilibrium models with the robust or recursive utility specifications that we shall study here.

Let C denote consumption and $\widehat{C}$ the logarithm of consumption. Suppose that the logarithm of consumption evolves as:

$$\widehat{C}_{t+1}(\mathsf{q}) - \widehat{C}_t(\mathsf{q}) = \kappa(X_t, \mathsf{q}W_{t+1}, \mathsf{q}).$$

Approximate this process by:

$$\widehat{C}_{t+1}(\mathsf{q}) - \widehat{C}_t(\mathsf{q}) \approx \widehat{C}^0_{t+1} - \widehat{C}^0_t + \mathsf{q}\left(\widehat{C}^1_{t+1} - \widehat{C}^1_t\right) + \frac{\mathsf{q}^2}{2}\left(\widehat{C}^2_{t+1} - \widehat{C}^2_t\right) \tag{11.16}$$

where

$$\widehat{C}^0_{t+1} - \widehat{C}^0_t = \kappa(X^0_t, 0, 0) := \eta^c_0$$

$$\widehat{C}^1_{t+1} - \widehat{C}^1_t = \kappa_{x'} X^1_t + \kappa_{w'} W_{t+1} + \kappa_q$$

$$\begin{aligned}\widehat{C}^2_{t+1} - \widehat{C}^2_t = {} & \kappa_{x'} X^2_t + {X^1_t}' \kappa_{xx'} X^1_t + 2\kappa_{qx'} X^1_t + \kappa_{qq} \\ & + 2{X^1_t}' \kappa_{xw'} W_{t+1} + {W_{t+1}}' \kappa_{ww'} W_{t+1} + 2\kappa_{qw'} W_{t+1}.\end{aligned}$$

In models with endogenous investment and savings, consumption dynamics as well as some of the state dynamics emerge as equilibrium outcomes. We use the approximating processes (11.13) and (11.16) as inputs for constructing an approximate continuation value process and its risk-adjusted counterpart under recursive utility preferences.

11.5. Incorporating preferences with enhanced uncertainty concerns

To approximate the recursive utility process, we deviate from common practice in macroeconomics by letting the risk aversion or robustness parameters in preferences depend on q:

$$\xi = \mathsf{q}\xi_o \qquad \gamma - 1 = \frac{\gamma_o - 1}{\mathsf{q}}$$

The aversion to model misspecification or the aversion to risk moves inversely with the parameter **q** when we embed the model of interest within a parameterized family of models. In effect, the variable **q** is doing double duty. Reducing **q** > 0 limits the overall exposure of the economy to the underlying shocks. This is offset by letting the preferences include a greater aversion to uncertainty. This choice of any expansion protocol has significant and enlightening consequences for continuation value processes and for the minimizing N process used to alter expectations. It allows for limiting behavior that expresses uncertainty implications at lower orders of approximation. It has antecedents in the control theory literature, and it has the virtue that implied uncertainty adjustments occur more prominently at lower-order terms in the approximation.

11.5.1. Order-zero

Write the order-zero expansion of (11.3) as

$$\widehat{V}_t^0 = \frac{1}{1-\rho} \log \left[(1-\beta) \exp[(1-\rho)\widehat{C}_t^0] + \beta \exp \left[(1-\rho) \widehat{R}_t^0 \right] \right]$$

$$\widehat{R}_t^0 = \widehat{V}_{t+1}^0,$$

where the second equation follows from noting that randomness vanishes in the limit as **q** approaches 0.

For order zero, write the consumption growth-rate process as

$$\widehat{C}_{t+1}^0 - \widehat{C}_t^0 = \eta_c^0.$$

The order-zero approximation of (11.3) is:

$$\widehat{V}_t^0 - \widehat{C}_t^0 = \frac{1}{1-\rho} \log \left[(1-\beta) + \beta \exp \left[(1-\rho) \left(\widehat{V}_{t+1}^0 - \widehat{C}_{t+1}^0 + \eta_c^0 \right) \right] \right]$$

We guess that $\widehat{V}_t^0 - \widehat{C}_t^0 = \eta_{v-c}^0$ and will have verified the guess once we solve:

$$\exp \left[(1-\rho) \left(\eta_{v-c}^0 \right) \right] = (1-\beta) + \beta \exp \left[(1-\rho) \left(\eta_{v-c}^0 \right) \right] \exp \left[(1-\rho) \eta_c^0 \right].$$

This equation implies

$$\exp \left[(1-\rho) \left(\eta_{v-c}^0 \right) \right] = \frac{1-\beta}{1-\beta \exp \left[(1-\rho) \eta_c^0 \right]}. \tag{11.17}$$

Equation (11.17) determines η_{v-c}^0 as a function of η_c^0 and the preference parameters ρ, β, but not the risk aversion parameter γ or its robust counterpart ξ. Specifically,

$$\eta^0_{v-c} = \frac{\log(1-\beta) - \log\left(1 - \beta \exp\left[(1-\rho)\eta^0_c\right]\right)}{1-\rho} \tag{11.18}$$

In the limiting $\rho = 1$ case,

$$\eta^0_{v-c} = \left(\frac{\beta}{1-\beta}\right)\eta^0_c.$$

11.5.2. Order-one

We temporarily take $\widehat{R}^1_t - \widehat{C}^1_t$ as given. We construct a first-order approximation to the nonlinear utility recursion (11.3)

$$\widehat{V}^1_t - \widehat{C}^1_t = \lambda\left(\widehat{R}^1_t - \widehat{C}^1_t\right) \tag{11.19}$$

where

$$\begin{aligned}
\lambda =& \left[\frac{\beta \exp\left[(1-\rho)\left(\eta^0_{v-c} + \eta^0_c\right)\right]}{(1-\beta) + \beta\exp\left[(1-\rho)\left(\eta^0_{v-c}+\eta^0_c\right)\right]}\right] \\
=& \left[\frac{\beta\exp\left[(1-\rho)\eta^0_c\right]}{(1-\beta)\exp\left[-(1-\rho)\eta^0_{v-c}\right] + \beta\exp\left[(1-\rho)\eta^0_c\right]}\right] \\
=& \left[\frac{\beta\exp\left[(1-\rho)\eta^0_c\right]}{1 - \beta\exp\left[(1-\rho)\eta^0_c\right] + \beta\exp\left[(1-\rho)\eta^0_c\right]}\right] \\
=& \beta\exp\left[(1-\rho)\eta^0_c\right].
\end{aligned}$$

Notice how the parameter ρ influences the weight λ when $\eta^0_c \neq 0$, in which case the log consumption process displays growth or decay. We will subsequently require that $\lambda < 1$. This restricts the subjective discount rate, $-\log\beta$, relative to the consumption growth rate η^0_c since

$$(1-\rho)\eta^0_c < -\log\beta.$$

When $\rho < 1$, the subjective discount rate has a positive lower bound in contrast to the case in which $\rho \geq 1$.

We next compute $\widehat{R}^1_t - \widehat{C}^1_t$. To facilitate this calculation, we construct:

$$\widetilde{V}_t \stackrel{\text{def}}{=} \frac{\widehat{V}_t - \widehat{V}_t^0}{\mathsf{q}} \tag{11.20}$$

$$\widetilde{R}_t \stackrel{\text{def}}{=} \frac{\widehat{R}_t - \widehat{V}_{t+1}^0}{\mathsf{q}}, \tag{11.21}$$

As q declines to zero, the numerators and denominators of right side of these constructions both go to zero. Their limits as q declines to zero turn out to be well defined, with limits denoted by $\widetilde{V}_t^0$ and $\widetilde{R}_t^0$. Importantly, from (11.4)

$$\widetilde{R}_t = \left(\frac{1}{1-\gamma_o}\right) \log \mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}\right] \mid \mathfrak{A}_t\right)$$

where we use the fact that $\widehat{V}_{t+1}^0 t$ is known at date t and $(1-\gamma)\mathsf{q} = 1-\gamma_o$. Taking limits as q declines to zero:

$$\widetilde{R}_t^0 = \left(\frac{1}{1-\gamma_o}\right) \log \mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}^0\right] \mid \mathfrak{A}_t\right). \tag{11.22}$$

Observe that

$$\begin{aligned} \widetilde{R}_t^0 &= \widehat{R}_t^1 \\ \widetilde{V}_t^0 &= \widehat{V}_t^1. \end{aligned}$$

Substituting these relations into (11.22) and subtracting $\widehat{C}_t^1$ from both sides results in:

$$\begin{aligned} &\widehat{R}_t^1 - \widehat{C}_t^1 = \\ &\left(\frac{1}{1-\gamma_o}\right) \log \mathbb{E}\left(\exp\left[(1-\gamma_o)\left[\left(\widehat{V}_{t+1}^1 - \widehat{C}_{t+1}^1\right) + \left(\widehat{C}_{t+1}^1 - \widehat{C}_t^1\right)\right]\right] \mid \mathfrak{A}_t\right). \end{aligned} \tag{11.23}$$

Finally, substituting formula (11.23) into the right side of (11.19) gives the recursion for the first-order continuation value:

$$\begin{aligned} &\widehat{V}_t^1 - \widehat{C}_t^1 = \\ &\left(\frac{\lambda}{1-\gamma_o}\right) \log \mathbb{E}\left(\exp\left[(1-\gamma_o)\left[\left(\widehat{V}_{t+1}^1 - \widehat{C}_{t+1}^1\right) + \left(\widehat{C}_{t+1}^1 - \widehat{C}_t^1\right)\right]\right] \mid \mathfrak{A}_t\right). \end{aligned} \tag{11.24}$$

This equation has a solution of the form:

$$\widehat{V}_t^1 - \widehat{C}_t^1 = {\upsilon_1}' X_t^1 + \upsilon_0, \tag{11.25}$$

which we can solve by "guess and verify".

Remark 11.5

To construct a solution for $\widehat{V}_t^1 - \widehat{C}_t^1$, conjecture a solution of the form (11.25). It follows from (11.24) that

$$\upsilon_1 = \lambda\left((\psi_{x'})'\upsilon_1 + \kappa_x\right), \upsilon_0 = \lambda\left(\upsilon_0 + {\upsilon_1}'\psi_q + \kappa_q + \frac{(1-\gamma_o)}{2}\left|{\upsilon_1}'\psi_{w'} + \kappa_{w'}\right|^2\right). \tag{11.26}$$

Deduce the second equation by observing that

$$\exp\left[(1-\gamma_o)\left(\widehat{V}_{t+1}^1 - \widehat{C}_{t+1}^1\right) + (1-\gamma_o)\left(\widehat{C}_{t+1}^1 - \widehat{C}_t^1\right)\right]$$

is distributed as a log normal. The solutions to equations (11.26) are:

$$\upsilon_1 = \lambda(I - \lambda\psi_{x'})^{-1}\kappa_{x'}, \upsilon_0 = \frac{\lambda}{(1-\lambda)}\left({\upsilon_1}'\psi_q + \kappa_q\right) + \frac{\lambda(1-\gamma_o)}{2(1-\lambda)}\left|{\upsilon_1}'\psi_{w'} + \kappa_{w'}\right|^2.$$

The continuation value has two components. The first is:

$${\upsilon_1}' X_t^1 + \frac{\lambda}{(1-\lambda)}\left({\upsilon_1}'\psi_q + \kappa_q\right) = \mathbb{E}\left[\sum_{j=1}^{\infty}\lambda^j\left(\widehat{C}_{t+j}^1 - \widehat{C}_{t+j-1}^1\right) \mid \mathfrak{A}_t\right]$$

and the second component is a constant long-run risk adjustment given by:

$$\frac{\lambda(1-\gamma_o)}{2(1-\lambda)}\left|{\upsilon_1}'\psi_{w'} + \kappa_{w'}\right|^2.$$

This second term is the variance of

$$\mathbb{E}\left[\sum_{j=1}^{\infty}\lambda^j\left(\widehat{C}_{t+j}^1 - \widehat{C}_{t+j-1}^1\right) \mid \mathfrak{A}_{t+1}\right] = (1-\lambda)\mathbb{E}\left[\sum_{j=1}^{\infty}\lambda^j\left(\widehat{C}_{t+j}^1 - \widehat{C}_t^1\right) \mid \mathfrak{A}_{t+1}\right] \tag{11.27}$$

conditioned on $\mathfrak{A}_t$ scaled by $\frac{\lambda(1-\gamma_o)}{2(1-\lambda)}$.

Remark 11.6

The formula for υ_1 depends on the parameter ρ. Moreover, υ_1 has a well-defined limit as λ tends to unity as does the variance of (11.27). This limiting variance:

$$\lim_{\lambda \to 1} \left| {\upsilon_1}' \psi_{w'} + \kappa_{w'} \right|^2.$$

converges to the variance of the martingale increment of $\widehat{C}^1$.

Remark 11.7

Consider the logarithm of the uncertainty-adjusted continuation value approximated to the first order. Note that from (11.25),

$$\widehat{V}^1_{t+1} - \widehat{C}^1_t = {\upsilon_1}' X^1_{t+1} + \upsilon_0 + \kappa_{x'} X^1_t + \kappa_{w'} W_{t+1}.$$

Substitute this expression into formula (11.23) and use the formula for the mean of random variable distributed as a log normal to show that

$$\widehat{V}^1_{t+1} - \widehat{R}^1_t = ({\upsilon_1}' \psi_{w'} + \kappa_{w'}) W_{t+1} - \left(\frac{1-\gamma_o}{2} \right) |{\upsilon_1}' \psi_w' + \kappa_{w'}|^2.$$

Associated with the first-order approximation, we construct:

$$N^0_{t+1} \stackrel{\text{def}}{=} \frac{\exp\left[(1-\gamma_o)\widetilde{V}^0_{t+1}\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}^0_{t+1}\right] \mid \mathfrak{A}_t\right)} = \frac{\exp\left[(1-\gamma_o)\widehat{V}^1_{t+1}\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widehat{V}^1_{t+1}\right] \mid \mathfrak{A}_t\right)}.$$

Equation (11.23) is a standard risk-sensitive recursion applied to log-linear dynamics. For instance, see [Tallarini, 2000]'s paper on risk-sensitive business cycles and [Hansen *et al.*, 2008]'s paper on measurement and inference challenges created by the presence of long-term risk.[4] Both of those papers assumed a logarithmic one-period utility function, so that for them $\rho = 1$. Here we have instead obtained the recursion as a first-order approximation without necessarily assuming log utility. Allowing for ρ to be different from one shows up in both the order zero and order one approximations, as reflected in (11.18) and (11.24), respectively. In accordance with (11.24), for the first-order approximation the parameter $\lambda = \beta$ when $\rho = 1$. But otherwise, it is different. Equation (11.23) also is very similar to a first-order approximation proposed in [Restoy and Weil, 2011]. Like formula (11.23), [Restoy and Weil, 2011] allow for $\rho \neq 1$. In contrast, our equation has an explicit constant term coming from the uncertainty adjustment, and we have an explicit formula for λ that depends on preference parameters and the consumption growth rate.

Remark 11.8

The calculation reported in Remark 11.7 implies that

$$\log N^0_{t+1} = (1-\gamma_o)\left(\widehat{V}^1_{t+1} - \widehat{R}^1_t\right) = (1-\gamma_o)\left({\upsilon_1}'\psi_{w'} + \kappa_{w'}\right)W_{t+1} - \frac{(1-\gamma_o)^2}{2}|{\upsilon_1}'\psi_{w'} + \kappa_{w'}|^2$$

As a consequence, under the change in probability measure induced by N^0_{t+1}, W_{t+1} has a mean given by

$$\begin{aligned}\mu^0 &\stackrel{\text{def}}{=} (1-\gamma_o)\left({\upsilon_1}'\psi_{w'} + \kappa_{w'}\right)' \\ &= \left(\frac{1}{\xi_o}\right)\left({\upsilon_1}'\psi_{w'} + \kappa_{w'}\right)'\end{aligned}$$

and with the same covariance matrix given by the identity. This is an approximation to robustness adjustment expressed as an altered distribution of the underlying shocks. It depends on $\gamma_o - 1 = \frac{1}{\xi_o}$ as well as the state dynamics as reflected by υ_1 and by the shock exposure vectors $\psi_{w'}$ and $\kappa_{w'}$. As we will see, this change of measure plays a role in the higher-order approximation, but it also gives a low-order representation of the implied shadow or market one-period compensation for exposure to uncertainty. It captures the following insight from “long-run risk” models: investor concerns about *long-term uncertainty* impacts *short-term asset valuation.* In contrast to the “long-run risk” literature, our analysis opens the door to a different interpretation. Instead of aversion to risk, it reflects an aversion to the misspecification of models or simplified perspectives on macroeconomic dynamics.

As we noted in Remark 11.6, $({\upsilon_1}'\psi_{w'} + \kappa_{w'})W_{t+1}$ is approximately the martingale component of the logarithm of consumption when λ is close to one. In Section 5.2 we showed that the variance of this component is challenging to estimate, a point originally made by [Hansen *et al.*, 2008]. This finding is part of the reason that we find it important to step back from rational expectations and limit investors confidence in the models they use for decision making.

11.5.3. Order two

Differentiating equation (11.3) a second time gives:

$$\widehat{V}^2_t = (1-\lambda)\widehat{C}^2_t + \lambda\widehat{R}^2_t + (1-\rho)(1-\lambda)\lambda\left(\widehat{R}^1_t - \widehat{C}^1_t\right)^2.$$

Equivalently,

$$\widehat{V}_t^2 - \widehat{C}_t^2 = \lambda \left(\widehat{R}_t^2 - \widehat{C}_t^2\right) + (1-\rho)(1-\lambda)\lambda\left(\widehat{R}_t^1 - \widehat{C}_t^1\right)^2. \tag{11.28}$$

Rewrite transformations (11.20) and (11.21) as

$$\mathsf{q}\widetilde{V}_t = \widehat{V}_t - \widehat{V}_t^0$$

$$\mathsf{q}\widetilde{R}_t = \widehat{R}_t - \widehat{V}_{t+1}^0.$$

Differentiating twice with respect to q and evaluating at $\mathsf{q} = 0$ gives:

$$2\frac{d}{d\mathsf{q}}\widetilde{V}_t + \mathsf{q}\frac{d^2}{d\mathsf{q}^2}\widetilde{V}_t\bigg|_{\mathsf{q}=0} = 2\widetilde{V}_t^1 = \widehat{V}_t^2.$$

$$2\frac{d}{d\mathsf{q}}\widetilde{R}_t + \mathsf{q}\frac{d^2}{d\mathsf{q}^2}\widetilde{R}_t\bigg|_{\mathsf{q}=0} = 2\widetilde{R}_t^1 = \widehat{R}_t^2.$$

Differentiating (11.21) with respect to q gives:

$$\frac{d\widetilde{R}_t}{d\mathsf{q}} = \frac{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}\right]\frac{d\widetilde{V}_{t+1}}{d\mathsf{q}} \mid \mathfrak{A}_t\right)}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}\right] \mid \mathfrak{A}_t\right)}.$$

and thus

$$\widehat{R}_t^2 = 2\widetilde{R}_t^1 = 2\mathbb{E}\left(N_{t+1}^0\widetilde{V}_{t+1}^1 \mid \mathfrak{A}_t\right) = \mathbb{E}\left(N_{t+1}^0\widehat{V}_{t+1}^2 \mid \mathfrak{A}_t\right), \tag{11.29}$$

where subtracting $\widehat{C}_t^2$ from $\widehat{R}_t^2$ gives:

$$\widehat{R}_t^2 - \widehat{C}_t^2 = \mathbb{E}\left[N_{t+1}^0\left(\widehat{V}_{t+1}^2 - \widehat{C}_{t+1}^2\right) + \left(\widehat{C}_{t+1}^2 - \widehat{C}_t^2\right) \mid \mathfrak{A}_t\right]. \tag{11.30}$$

Substituting this formula into (11.29) gives:

$$\begin{aligned}\widehat{V}_t^2 - \widehat{C}_t^2 = \lambda\mathbb{E}\left(N_{t+1}^0\left[\left(\widehat{V}_{t+1}^2 - \widehat{C}_{t+1}^2\right) + \left(\widehat{C}_{t+1}^2 - \widehat{C}_t^2\right)\right] \mid \mathfrak{A}_t\right) \\ + (1-\rho)(1-\lambda)\lambda\left(\widehat{R}_t^1 - \widehat{C}_t^1\right)^2.\end{aligned} \tag{11.31}$$

Even if the second-order contribution to the consumption process is zero, there will be nontrivial adjustment to the approximation of $\widehat{V} - \widehat{C}$ because $\left(\widehat{R}^1 - \widehat{C}^1\right)^2$ is different from zero. This term vanishes when $\rho = 1$, and its sign will be different depending on whether ρ is bigger or smaller than one.

11.6. Stochastic discount factor approximation

We approximate $\left[\widehat{S}_{t+1} - \widehat{S}_t\right]$ in formula (11.8) as

$$\widehat{S}_{t+1} - \widehat{S}_t \approx \left[\widehat{S}^0_{t+1} - \widehat{S}^0_t\right] + \left[\widehat{S}^1_{t+1} - \widehat{S}^1_t\right] + \frac{1}{2}\left[\widehat{S}^2_{t+1} - \widehat{S}^2_t\right]$$

where

$$\widehat{S}^0_{t+1} - \widehat{S}^0_t \overset{\text{def}}{=} \log\beta - \rho\eta^0_c$$

$$\widehat{S}^1_{t+1} - \widehat{S}^1_t \overset{\text{def}}{=} -\widehat{C}^1_{t+1} + \widehat{C}^1_t + (\rho - 1)\left(\widehat{V}^1_{t+1} - \widehat{C}^1_{t+1}\right) - (\rho - 1)\left(\widehat{R}^1_t - \widehat{C}^1_t\right)$$

$$\widehat{S}^2_{t+1} - \widehat{S}^2_t \overset{\text{def}}{=} -\widehat{C}^2_{t+1} + \widehat{C}^2_t + (\rho - 1)\left(\widehat{V}^2_{t+1} - \widehat{C}^2_{t+1}\right) - (\rho - 1)\left(\widehat{R}^2_t - \widehat{C}^2_t\right)$$

We now consider two different approaches to approximating N^*_{t+1}.

11.6.1. Approach 1

Write

$$N^*_{t+1} = \frac{\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}\right] \mid \mathfrak{A}_t\right)}$$

$$= \frac{\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}\right]}{\exp\left[(1-\gamma_o)\widetilde{R}_t\right]}$$

Form the ``first-order'' approximation:

$$\log N^*_{t+1} \approx (1-\gamma_o)\left[\left(\widetilde{V}^0_{t+1} - \widetilde{R}^0_t\right) + \mathsf{q}\left(\widetilde{V}^1_{t+1} - \widetilde{R}^1_t\right)\right]$$

$$= (1-\gamma_o)\left[\left(\widehat{V}^1_{t+1} - \widehat{R}^1_t\right) + \frac{\mathsf{q}}{2}\left(\widehat{V}^2_{t+1} - \widehat{R}^2_t\right)\right]$$

We combine a first-order approximation of $\log N^*_{t+1}$ with a second-order approximation of $\widehat{S}_{t+1} - \widehat{S}_t$:

$$\log S_{t+1} - \log S_t \approx (1-\gamma_o)\left[\left(\widehat{V}^1_{t+1} - \widehat{R}^1_t\right) + \frac{1}{2}\left(\widehat{V}^2_{t+1} - \widehat{R}^2_t\right)\right]$$

$$+ \left(\widehat{S}^0_{t+1} - \widehat{S}^0_t\right) + \left(\widehat{S}^1_{t+1} - \widehat{S}^1_t\right) + \frac{1}{2}\left(\widehat{S}^2_{t+1} - \widehat{S}^2_t\right)$$

which preserves the quadratic approximation of $\log S_{t+1} - \log S_t$. Note that If we were to use a second-order approximation of N^*_{t+1}, it would push us outside the class of exponentially quadratic stochastic discount factors.

11.6.2. Approach 2

Next consider an alternative modification of Approach 1 whereby:

$$\begin{aligned} N^*_{t+1} &\approx \frac{\exp\left[(1-\gamma_o)\left(\widetilde{V}^0_{t+1} + \widetilde{V}^1_{t+1}\right)\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\left(\widetilde{V}^0_{t+1} + \widetilde{V}^1_{t+1}\right)\right] \mid \mathfrak{A}_t\right)} \\ &= \frac{\exp\left[(1-\gamma_o)\left[\left(\widehat{V}^1_{t+1} - \widehat{R}^1_t\right) + \frac{1}{2}\left(\widehat{V}^2_{t+1} - \widehat{R}^2_t\right)\right]\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\left[\left(\widehat{V}^1_{t+1} - \widehat{R}^1_t\right) + \frac{1}{2}\left(\widehat{V}^2_{t+1} - \widehat{R}^2_t\right)\right]\right] \mid \mathfrak{A}_t\right)} \\ &\overset{\text{def}}{=} \widetilde{N}_{t+1} \end{aligned} \tag{11.32}$$

and $\log \widetilde{N}_t$ is used in conjunction with

$$\left(\widehat{S}^0_{t+1} - \widehat{S}^0_t\right) + \left(\widehat{S}^1_{t+1} - \widehat{S}^1_t\right) + \frac{1}{2}\left(\widehat{S}^2_{t+1} - \widehat{S}^2_t\right).$$

By design, this approximation of N^*_{t+1} will have conditional expectation equal to one, in contrast to the approximation used in Approach 1. With a little bit of algebraic manipulation, it can be shown that this approximation induces a distributional change for W_{t+1} with a conditional mean that is affine in X_t and an altered conditional variance matrix that is constant over time.

To better understand this choice of approximation, consider the family of random variables (indexed by q)

$$(1-\gamma_o)\left(\widetilde{V}^0_{t+1} + \mathsf{q}\widetilde{V}^1_{t+1}\right) - \log \mathbb{E}\left(\exp\left[(1-\gamma_o)\left(\widetilde{V}^0_{t+1} + \mathsf{q}\widetilde{V}^1_{t+1}\right)\right] \mid \mathfrak{A}_t\right).$$

The corresponding family of exponentials has conditional expectation one and the $\mathsf{q} = 1$ member is the proposed approximation for N^*_{t+1}. Differentiate the family with respect to q:

$$\widetilde{V}^1_{t+1} - \frac{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}^0_{t+1}\right]\widetilde{V}^1_{t+1} \mid \mathfrak{A}_t\right)}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}^0_{t+1}\right] \mid \mathfrak{A}_t\right)} = \widetilde{V}^1_{t+1} - \widetilde{R}^1_t.$$

Thus this family of random variables has the same first-order approximation in q as the one we derived previously for $\log N^*_{t+1}$.

As a change of probability measure, this approximation will induce state dependence in the conditional mean and will alter the covariance matrix of the shock vector. We find this approach interesting because it links back directly to the outcome of the robustness formulation we described in Chapter 9.

11.7. A planner's problem with recursive utility

The [Bansal and Yaron, 2004] example, along with many others that build connections between the macro economy and asset value, takes aggregate consumption as pre-specified. As we open the door to a richer collection of macroeconomic models, it becomes important to entertain more endogeneity, including investment and other variables familiar to macroeconomics.

Write a triangular system with stochastic growth as:

$$\begin{aligned} X_{t+1}(\mathsf{q}) &= \psi^x\left[D_t(\mathsf{q}), X_t(\mathsf{q}), \mathsf{q}W_{t+1}, \mathsf{q}\right] \\ \log G_{t+1}(\mathsf{q}) - \log G_t(\mathsf{q}) &= \psi^g\left[D_t(\mathsf{q}), X_t(\mathsf{q},), \mathsf{q}W_{t+1}, \mathsf{q}\right], \end{aligned} \tag{11.33}$$

where D_t is a date t decision vector for the planner. Define $\widehat{G}_t = \log G_t$. In addition, we impose

$$\begin{aligned} 0 &= \phi[D_t(\mathsf{q}), X_t(\mathsf{q})] \\ \widehat{C}_t(\mathsf{q}) &= \kappa\left[D_t(\mathsf{q}), X_t(\mathsf{q})\right] + \widehat{G}_t(\mathsf{q}). \end{aligned} \tag{11.34}$$

where the first equation is a vector of static constraints and the second constructs the measure of consumption that enters preferences.

We extend the approximations by using a co-state formulation. There are two essentially equivalent interpretations of these co-states. One is they function as a set of Lagrange multipliers on the state evolution equations. The other is that they are partial derivatives of value functions. The co-state equations are forward-looking, linking next period's co-state vector to this period's co-state vector. Given the recursive utility structure, we must include the implied value functions in the computations as they enter the relations of interest.

The first-order conditions for D are:

$$
\begin{aligned}
&(1-\beta)\exp\left[(1-\rho)\left(\widehat{C}_t-\widehat{V}_t\right)\right]\kappa_d(D_t,X_t)+\phi_{d'}(D_t,X_t)'MS_t\\
&+\beta\exp\left[(1-\rho)\left(\widehat{R}_t-\widehat{V}_t\right)\right]\mathbb{E}\left(N^*_{t+1}\psi^x_{d'}(D_t,X_t,\mathsf{q}W_{t+1},\mathsf{q})'MX_{t+1}\mid\mathfrak{A}_t\right)\\
&+\beta\exp\left[(1-\rho)\left(\widehat{R}_t-\widehat{V}_t\right)\right]\mathbb{E}\left(N^*_{t+1}\psi^g_{d'}(D_t,X_t,\mathsf{q}W_{t+1},\mathsf{q})'MG_{t+1}\mid\mathfrak{A}_t\right)\\
&=0.
\end{aligned}
\tag{11.35}
$$

where MX_{t+1} and MG_{t+1} are the co-states, or implicit multipliers, one for each the state evolution, and MS_t is a multiplier on the first static constraint in (11.34). Recall that $N^*_{t+1}=\exp\left[(1-\gamma)\left(\widehat{V}_{t+1}-\widehat{R}_t\right)\right]$ used for making an uncertainty adjustment in valuation.

In addition, we solve a forward-looking co-state equation given by

$$
\begin{aligned}
&\begin{bmatrix}MX_t\\MG_t\end{bmatrix}=(1-\beta)\exp\left[(1-\rho)\left(\widehat{C}_t-\widehat{V}_t\right)\right]\begin{bmatrix}\kappa_x(D_t,X_t)\\1\end{bmatrix}+\begin{bmatrix}\phi_x(D_t,X_t)'MS_t\\0\end{bmatrix}\\
&+\beta\exp\left[(1-\rho)\left(\widehat{R}_t-\widehat{V}_t\right)\right]\times\\
&\mathbb{E}\left(N^*_{t+1}\begin{bmatrix}\psi^x_{x'}(D_t,X_t,\mathsf{q}W_{t+1},\mathsf{q})' & \psi^g_{x'}(D_t,X_t,\mathsf{q}W_{t+1},\mathsf{q})'\\0 & 1\end{bmatrix}\begin{bmatrix}MX_{t+1}\\MG_{t+1}\end{bmatrix}\mid\mathfrak{A}_t\right).
\end{aligned}
\tag{11.36}
$$

Approximation formulas, (11.23), (11.24), (11.30), (11.31), that we deduced for $\widehat{V}_t-\widehat{C}_t$ and $\widehat{R}_t-\widehat{C}_t$ have immediate counterparts for $\widehat{V}_t-\widehat{G}_t$ and $\widehat{R}_t-\widehat{G}_t$, from which we can build an approximation of $\widehat{R}_t-\widehat{V}_t$.

From recursive utility updating equation:

$$
\exp\left[(1-\rho)\widehat{V}_t\right]=(1-\beta)\exp\left[(1-\rho)\widehat{C}_t\right]+\beta\exp\left[(1-\rho)\widehat{R}_t\right].
$$

Dividing both sides of the equation by $\exp\left[(1-\rho)\widehat{V}_t\right]$ gives

$$
1=(1-\beta)\exp\left[(1-\rho)\left(\widehat{C}_t-\widehat{V}_t\right)\right]+\beta\exp\left[(1-\rho)\left(\widehat{R}_t-\widehat{V}_t\right)\right].
$$

From the relation, we see that $MG_t=1$ for $t\geq 0$ satisfies the second block of co-state equation (11.36). In fact, this is the solution of interest.

11.7.1. An economy with long-run uncertainty

Consider an AK model with recursive utility and adjustment costs.

The exogenous state dynamic capture both long-run uncertainty in the mean growth rate and the overall volatility in the economy.

$$
\begin{aligned}
Z_{1,t+1} - Z_{1,t} = & -\nu_1 Z_{1,t} + \exp\left(\frac{1}{2} Z_{2,t}\right)\sigma_1 W_{t+1} \\
Z_{2,t+1} - Z_{2,t} = & -\nu_2 \left[1 - \mu_2 \exp\left(-Z_{2,t}\right)\right] \\
& -\frac{1}{2}|\sigma_2|^2 \exp\left(-Z_{2,t}\right) + \exp\left(-\frac{1}{2} Z_{2,t}\right)\sigma_2 W_{t+1}.
\end{aligned} \tag{11.37}
$$

The state variable, $Z_{2,t}$ is included to capture stochastic volatility. The discrete-time dynamics for $\{\exp(Z_{2,t})\}$ approximate a continuous-time version of what is called a square root process due to Feller. Let

$$
X_t = \begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix}.
$$

With these exogenous dynamics, we obtain the following zero and first-order approximations:

$$
Z^0_{1,t} = 0 \quad \exp\left(Z^0_{2,t}\right) = \mu_2,
$$

and

$$
\begin{aligned}
Z^1_{1,t+1} - Z^1_{1,t} = & -\nu_1 Z^1_{1,t} + \sqrt{\mu_2}\sigma_1 W_{t+1} \\
Z^1_{2,t+1} - Z^1_{2,t} = & -\nu_2 Z^1_{2,t} + \sqrt{\frac{1}{\mu_2}}\sigma_2 W_{t+1}.
\end{aligned}
$$

We impose the resource constraint:

$$
C_t + I_t = \alpha K_t.
$$

The endogenous state dynamics are given by:

$$
\begin{aligned}
\widehat{K}_{t+1} - \widehat{K}_t = & \left[\frac{1}{\zeta}\log\left(1 + \zeta\frac{I_t}{K_t}\right) + \nu_k Z_{1,t} - \iota_k\right] \\
& -\frac{1}{2}|\sigma_k|^2 \exp\left(Z_{2,t}\right) + \exp\left(\frac{1}{2} Z_{2,t}\right)\sigma_k W_{t+1}
\end{aligned}
$$

where $\widehat{K}_t = \log K_t = \widehat{G}_t$. The planner choice variable $D_t = \left(\frac{C_t}{K_t}, \frac{I_t}{K_t}\right)$. Rewrite the current-period resource constraints as:

$$
\begin{aligned}
0 &= \alpha - D_{1,t} - D_{2,t} \\
\widehat{C}_t - \widehat{G}_t &= \log D_{1,t}.
\end{aligned} \tag{11.38}
$$

Express the first-order conditions for the consumption-capital and investment-capital ratios as:

$$
\begin{aligned}
0 &= -MS_t + \frac{1-\beta}{D_{1,t}} \exp\left[(1-\rho)\left(\widehat{C}_t - \widehat{G}_t\right)\right] \\
0 &= -MS_t + \exp\left[(1-\rho)\left(\widehat{R}_t - \widehat{G}_t\right)\right]\left(\frac{\beta}{1+\zeta D_{2,t}}\right)
\end{aligned} \tag{11.39}
$$

It is convenient to rewrite the first-order conditions for the consumption-capital ratio as:

$$
\log MS_t + \log D_{1,t} = \log(1-\beta) + (1-\rho)\left(\widehat{C}_t - \widehat{G}_t\right),
$$

which in turn implies that

$$
\log MS_t = \log(1-\beta) - \rho\left(\widehat{C}_t - \widehat{G}_t\right).
$$

More generally, we will seek to approximate $\log MS_t$, as we expect the multiplier MS_t to be positive. Notice that these first-order conditions do not depend on the co-state process $\{MX_t : t \geq 0\}$. We can solve the planner's problem using (11.39) and up-dating the continuation value processes and its uncertainty-adjusted counterpart until convergence. When $\rho = 1$, $\widehat{R}_t - \widehat{G}_t$ drops out of the first-order conditions and both components of D_t are constant since

$$
MS_t = \frac{1}{D_{1,t}} = \frac{1}{\alpha - D_{2,t}}.
$$

When $\rho \neq 1$, D_t depends on the exogenous state X_t.

11.7.2. First-order approximation when $\rho = 1$

The first-order conditions for $D_{2,t}$ imply that:

$$
\frac{1-\beta}{D_{2,t} - \alpha} + \frac{\beta}{1+\zeta D_{2,t}} = 0.
$$

Solving for D gives:

$$
D^*_{2,t} = \frac{(\beta - 1) + \beta\alpha}{\beta + (1-\beta)\zeta},
$$

which is independent of the state, as should be expected since $\rho = 1$. From the capital evolution it follows from the order-zero approximation that

$$
\widehat{K}^0_{t+1} - \widehat{K}^0_t = \left[\frac{1}{\zeta}\log\left(1 + \zeta D^*_{2,t}\right) - \iota_k\right].
$$

The order one approximation is then:

$$
\widehat{K}^1_{t+1} - \widehat{K}^1_t = \nu_k Z^1_{1,t} + \sqrt{\mu_2}\sigma_k W_{t+1}.
$$

Stochastic volatility, as in the [Bansal and Yaron, 2004] model of consumption dynamics, will be present in the second-order approximation.

11.7.3. Second-order approximation when $\rho = 1$

We next consider the second-order approximations. The second-order approximation for $\{Z_{2,t}\}$ does not contribute to the planner's solution or to the implied shadow prices and thus we drop it from the analysis. For the remaining two state variables, we find that

$$
\begin{aligned}
Z^2_{1,t+1} - Z^2_{1,t} =& \; -\nu_1 Z^2_{1,t} - \mu_2|\sigma_1|^2 + \sqrt{\mu_2} Z^1_{2,t}\sigma_1 W_{t+1} \\
\widehat{K}^2_{t+1} - \widehat{K}^2_t =& \; -\mu_2|\sigma_k|^2 + \sqrt{\mu_2} Z^1_{2,t}\sigma_k W_{t+1}
\end{aligned}
$$

where we previously noted that $\{Z^1_{2,t}\}$ evolves as a first-order autoregression.

The combined approximation for $\mathbf{q} = 1$ uses:

$$
\begin{aligned}
Z_{1,t+1} - Z_{1,t} \approx & \; -\nu_1 Z_{1,t} \\
& -\frac{\mu_2|\sigma_1|^2}{2} + \sqrt{\mu_2}\sigma_1 W_{t+1} + \frac{\mu_2 Z_{2,t}}{2}\sigma_1 W_{t+1} \\
Z_{2,t+1} - Z_{2,t} \approx & \; -\nu_2 Z_{2,t} + \sqrt{\frac{1}{\mu_2}}\sigma_2 W_{t+1} \\
\widehat{K}_{t+1} - \widehat{K}_t \approx & \left[\frac{1}{\zeta}\log\left(1 + \zeta D^*\right) - \iota_k\right] + \nu_k Z_{1,t} \\
& -\frac{\mu_2|\sigma_k|^2}{2} + \sqrt{\mu_2}\sigma_k W_{t+1} + \frac{\mu_2 Z_{2,t}}{2}\sigma_k W_{t+1}.
\end{aligned}
$$

The approximate dynamics for the exogenous states remains the same for $\rho \neq 1$, but the solution for D^* becomes state dependent and the approximate dynamic evolution for capital is altered.

11.7.4. Shock elasticities

We use the shock elasticities to explore pricing implications of this recursive utility specification. In what follows, we use exponential/linear/quadratic implementation by [Borovička and Hansen, 2014] and by

[Borovička and Hansen, 2016] with the parameter configuration given in Table 1 of [Hansen *et al.*, 2024]. This latter reference combines inputs from other sources including [Schorfheide *et al.*, 2018] and [Hansen and Sargent, 2021].

Fig. 11.1 shows how the consumption elasticities vary with the preference parameters ρ and γ for the shock to the technology growth rate. In addition, Fig. 11.2 shows that the investment-capital ratio elasticities. When we vary ρ, we also change the productivity parameter α in order that the steady state growth rates remain the same as in the table that follows:

ρ	α
1	0.092
2/3	0.082
3/2	0.108

As can be seen from the figures, the parameter ρ has prominent impacts on the consumption exposure and investment-capital elasticities, but it only has a modest impact on the price elasticities. In contrast, the parameter γ has an important impact on the price elasticities, but it has virtually no impact on the quantity elasticities. In particular, Fig. 11.2 shows that the investment-capital ratio elasticities are highly sensitive to ρ, with the sign of the initial shocks determined by whether ρ is greater or less than one. Specifically, when $\rho < 1$ (intertemporal elasticity of substitution is greater than one), the initial investment responds positively to a shock to the productivity growth rate.

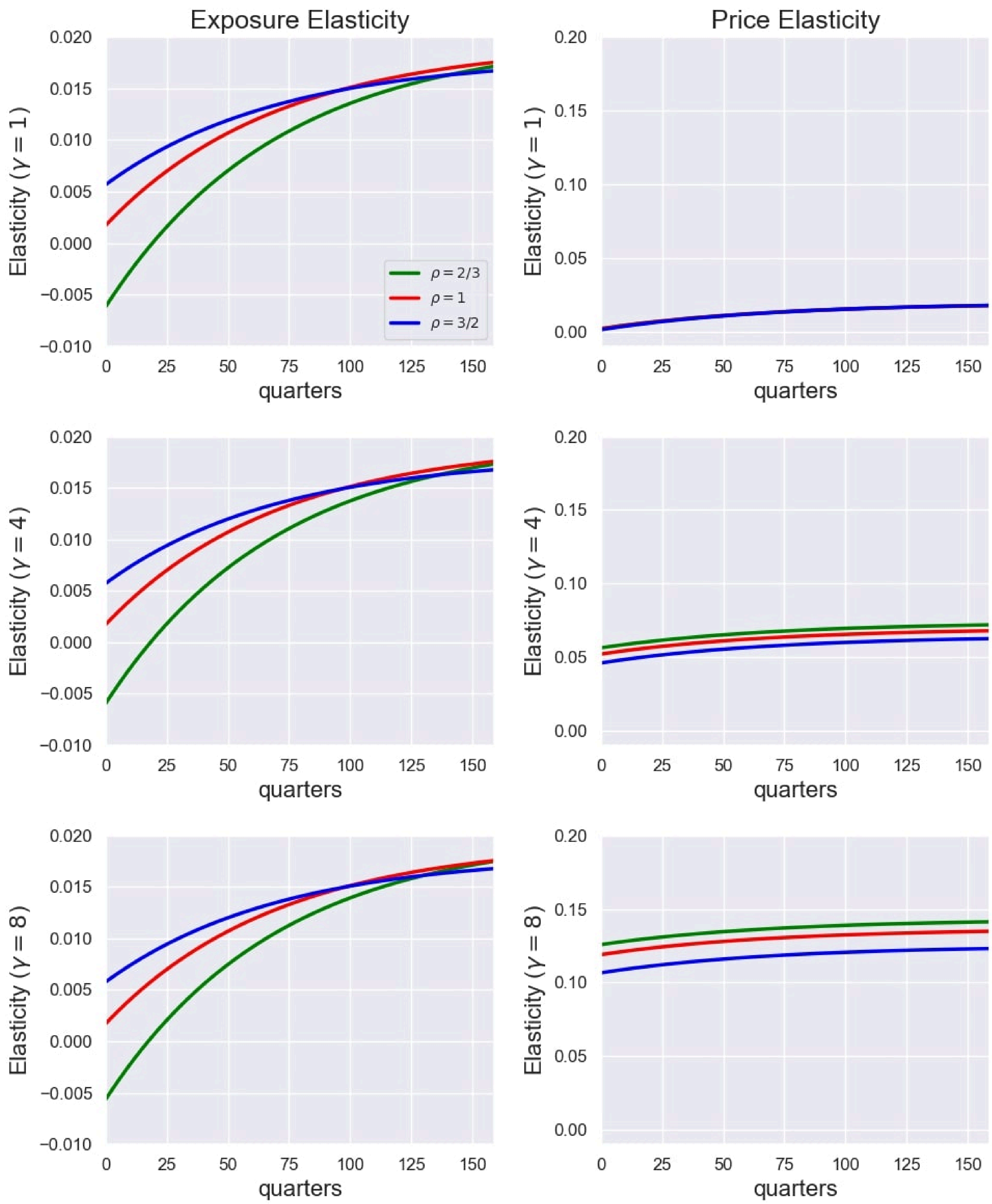

Fig. 11.1 Consumption shock-exposure and price elasticities for the growth-rate shock for different values of ρ and γ. The initial stochastic volatility state is set to its median. Each of the row panels reports computations for different values of γ.

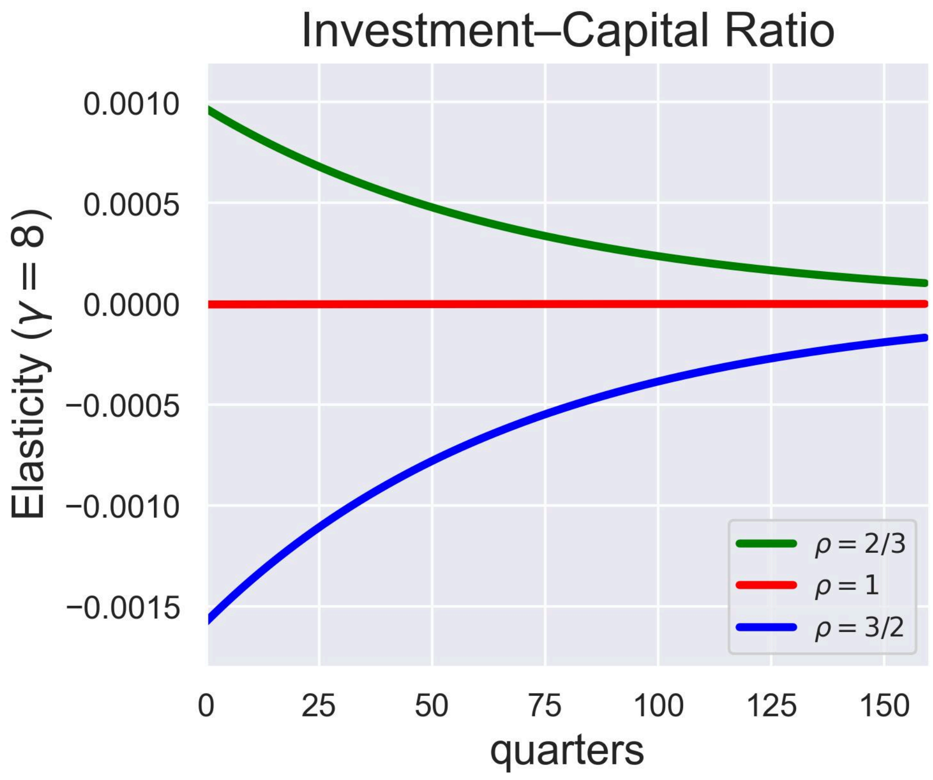

Fig. 11.2 Investment-capital exposure elasticities for the growth rate shock for various values of ρ. The initial stochastic volatility state is set to its median, and $\gamma = 8$.

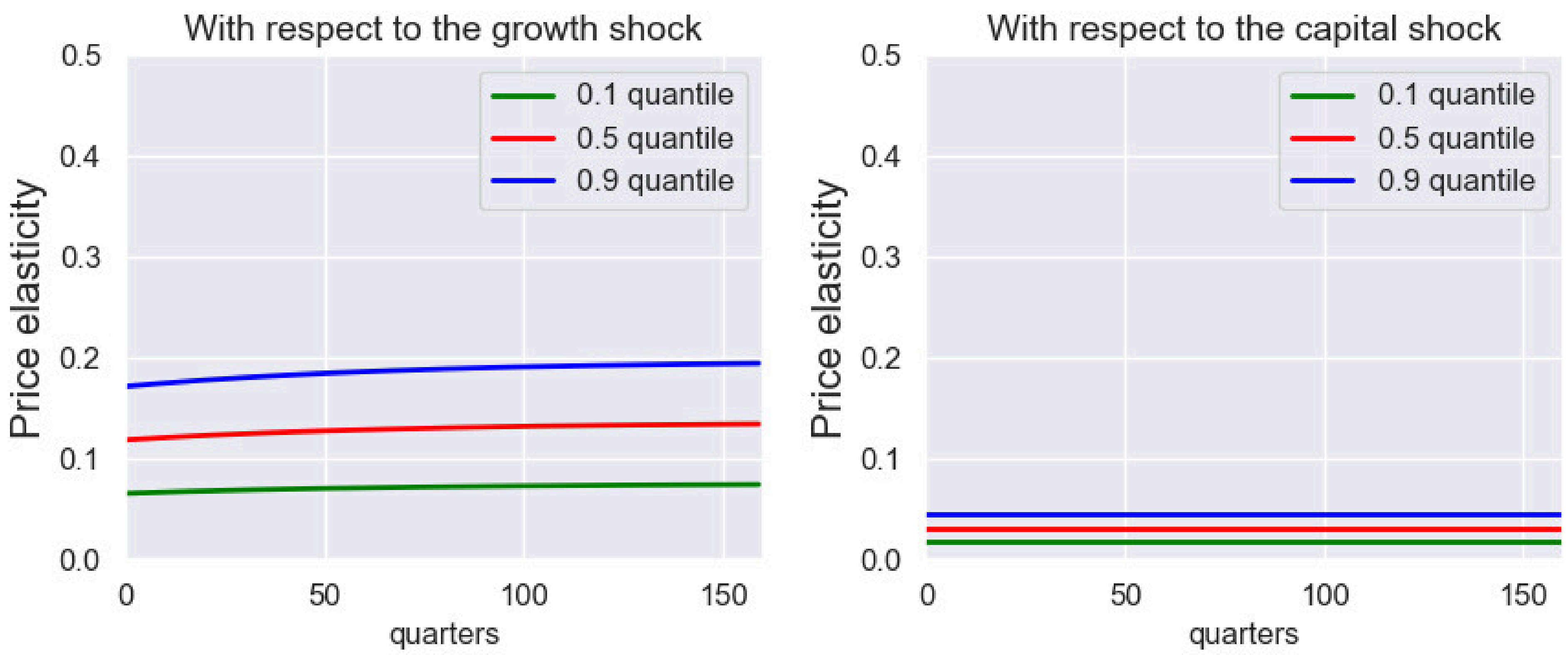

Fig. 11.3 Alternative stochastic volatility quantiles for the price elasticities for the growth and capital shocks. $\rho = 1, \gamma = 8, \beta = .99$.

Fig. 11.3 gives the shock price elasticities when $\rho = 1$ and $\gamma = 8$ for the growth rate shock and the capital shock. Stochastic volatility induces state dependence in these plots as reflected by the quantiles. Since the recursive utility preferences are forward-looking as reflected by the continuation-value contribution to the one-period increment to the stochastic discount factor process (see (11.8)), this forward-looking contribution is reflected in shock price elasticities that are substantially larger for the growth-rate shock.[5]. The primary contribution of stochastic volatility is to induce state dependence for the other elasticities as reflected by the quantiles. While the elasticities for the volatility shock are different from zero, their contribution is much smaller than the other shocks and thus are not reported in Fig. 11.3.

11.7.5. Another model of intertemporal substitution/complementarity.

We now extend the preference specification of the consumers in the AK model of Section 11.7.1 to explore implications of time nonseparability in preferences. We are motivated to do so by rather substantial previous literature. An important earlier contributor is [Ryder and Heal, 1973], solved a social planner's problem with stochastic growth. [Sundaresan, 1989], [Constantinides, 1990], [Heaton, 1995], and [Hansen *et al.*, 1999] consider asset pricing implications with *internal* habit persistence. These papers essentially explore decentralizations of the planner's problem. The latter paper considers simultaneously a recursive robustness specification as we will do here, although for a different model specification.

[Pollak, 1970] introduced a version of *external* habit persistence in the consumer demand function. The stock H_t enters preferences as a societal or external input and not recognizable as an individual input in the current time period. Thus there is an externality induced by this specification of preferences. [Abel, 1990] and [Campbell and Cochrane, 1999] explore implications of *external* habit persistence, along with many subsequent papers on asset pricing with *external* habit persistence. Much of the latter literature abstracts from production, but there are exceptions. See for instance, [Lettau and Uhlig, 2000]. The equilibrium solution with external habit persistence deviates from the planner problem because the investor is assumed not to internalize the consumption impact on habits. A benevolent planner would internalize the consumption impact on habits, so the equilibrium with the external habit specification is not socially optimal, opening the door to prudent policy interventions. In what follows we will compare the external and internal habit specifications, where we have dual interpretations of the internal habit version. Either the habits are internalized by the individual decision makers or it provides the socially efficient outcomes for an external habit specification. In the latter case, a prudent policy maker would aim to address these externalities.

Relative to our previous computations, we introduce an additional state variable, which we will call the habit stock, H_t. This stock evolves as:

$$H_{t+1} = \exp(-\nu_h) H_t + [1 - \exp(-\nu_h)] I_{h,t}.$$

where $\nu_h > 0$ is a depreciation rate. We think of this new investment, $I_{h,t}$, as measured consumption. Notice that the sum of the coefficients on the right side of the evolution equation sum to one. This allows us to interpret H_{t+1} as a geometric average of current and past values of measured consumption.

We write the evolution equation in logarithms as

$$\log H_{t+1} = \log\left[\exp(-\nu_h + \log H_t) + [1 - \exp(-\nu_h)] I_{h,t}\right].$$

For numerical purposes, we transform this equation to be:

$$
\begin{aligned}
&(\log H_{t+1} - \log K_{t+1}) \\
&= \log K_t - \log K_{t+1} + \log \left[\exp(-\nu_h + \log H_t - \log K_t) + [1 - \exp(-\nu_h)] \left(\frac{I_{h,t}}{K_t} \right) \right] \\
&= -\left[\frac{1}{\zeta} \log \left(1 + \zeta \frac{I_t}{K_t} \right) - \nu_k Z_{1,t} + \iota_k \right] + \frac{1}{2} |\sigma_k|^2 \exp\left(Z_{2,t} \right) - \exp\left(\frac{1}{2} Z_{2,t} \right) \sigma_k W_{t+\epsilon} \\
&\quad + \log \left[\exp(-\nu_h + \log H_t - \log K_t) + [1 - \exp(-\nu_h)] \left(\frac{I_{h,t}}{K_t} \right) \right],
\end{aligned}
$$

where we treat $\log H_t - \log K_t$ as one of the components of X_t and we substitute the evolution of $\log K_{t+1}$ to depict the evolution of this component. As previously, $\widehat{G}_t = \log K_t$.

We modify the output constraint to be:

$$
I_{h,t} + I_{k,t} = \alpha K_t
$$

where $I_{k,t}$ is the investment in the capital stock. Analogous to previous formulation, we transform this equation to be:

$$
\frac{I_{h,t}}{K_t} + \frac{I_{k,t}}{K_t} = \alpha
$$

where $\frac{I_{h,t}}{K_t}$ and $\frac{I_{k,t}}{K_t}$ are the two components of D_t.

What enters the utility function each date inside the recursive utility preference specification is the CES aggregate:

$$
C_t = \left[(1-\lambda)(I_{h,t})^{1-\tau} + \lambda (H_t)^{1-\tau} \right]^{\frac{1}{1-\tau}} \tag{11.40}
$$

for $\tau \geq 0$. This specification captures a form of intertemporal complementarity and intertemporal substitution in preferences. In static demand theory, $\tau = 0$ implies perfect substitutes, and when $\tau > 1$, the preferences display a form of complementarity as reflected in the cross price effects on demand. There is a different notion of intertemporal complementarity as originally defined by [Ryder and Heal, 1973]. Under this notion, increasing H_t, keeping $I_{h,t}$ fixed should decrease C_t. (They impose other restrictions.) This form of intertermporal complementarity can instead be captured by letting $\lambda < 0$. Effectively, H_t is a “bad'' not a “good,'' in the CES aggregator. Setting $0 < \lambda \leq 1$ captures a form of consumption durability.

For computational purposes, we divide both sides of (11.40) by K_t and take logarithms:

$$
\log C_t - \log K_t = \frac{1}{1-\tau} \log \left[(1-\lambda) \left(\frac{I_{h,t}}{K_t} \right)^{1-\tau} + \lambda \exp\left[(1-\tau) \left(\log H_t - \log K_t \right) \right] \right]
$$

Consistent with the habit-persistence literature, we allow the parameter λ to be negative. When τ is one, this becomes a Cobb-Douglas specification:

$$\log C_t - \log K_t = (1-\lambda)\left(\log I_{h,t} - \log K_t\right) + \lambda\left(\log H_t - \log K_t\right)$$

Remark 11.9

The asset pricing literature often features the computationally more challenging case in which $\tau = 0$ and $\lambda < 0$. The local approximation methods we describe here could provide a particularly poor approximation for this limiting parameter configuration. In what follows, we will entertain values of positive values of τ that are less than one.

For simplicity, consider the case in which $\rho = 1$. Recall that ρ also impacts intertemporal substitution in preferences so by setting it to one we feature the novel contribution coming from this dynamic specification of preferences. While we investigate other specifications of τ, for the moment suppose that $\tau = 1$. The first-order conditions for D_t are:

$$\begin{aligned}(1-\beta)\begin{bmatrix}\frac{1-\lambda}{D_{1,t}}\\ 0\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix} MS_t + \mathbb{E}\left[N^*_{t+1}\begin{bmatrix}\frac{\beta K_t[1-\exp(-\nu_h)]}{\exp(X_{t+1})K_{t+1}} & 0 & 0\\ \frac{1}{1+\zeta D_{2,t}} & 0 & 0\end{bmatrix} MX_{t+1} \mid \mathfrak{A}_t\right]\\ + \beta\begin{bmatrix}0\\ \frac{1}{1+\zeta D_{2,t}}\end{bmatrix}\mathbb{E}\left(N^*_{t+1} MG_{t+1} \mid \mathfrak{A}_t\right) = 0.\end{aligned}$$

From the first-order conditions for $D_{1,t}$, we find that

$$MS_t = (1-\beta)\left(\frac{1-\lambda}{D_{1,t}}\right) + \mathbb{E}\left(N^*_{t+1}\begin{bmatrix}\frac{\beta K_t[1-\exp(-\nu_h)]}{\exp(X_{t+1})K_{t+1}} & 0 & 0\end{bmatrix} MX_{t+1} \mid \mathfrak{A}_t\right). \tag{11.41}$$

This depicts the marginal value of MS_t in terms of both a current marginal utility contribution and a forward-looking piece coming from the planner internalizing the intertemporal contribution to preferences. When this contribution is not internalized, the expectation term is omitted from the first-order conditions.
We choose to use $I_{h,t}$ as the date t numeraire instead of C_t, as the former is measured consumption. In light of this choice, the logarithm of the one-period stochastic discount factor is

$$\widehat{S}_{t+1} - \widehat{S}_t = \log\beta + \log MS_{t+1} - \log MS_t + \widehat{K}_{t+1} - \widehat{K}_t.$$

In our calculations, we approximate $\log MS_t$ since it enters this formula for the logarithmic stochastic discount factor construction.

In the figures that follow we compare results for the internal specification versus the external specification of preferences, which differ in terms of how the investor views the intertemporal contribution to preferences. There are two ways to interpret these comparisons. One way is as a comparison of two specifications of investor preferences. The other is to presume that the intertemporal contribution to preferences is external from the standpoint of the investor, but what is referred to as internal is a planner solution that internalizes the externality via a policy intervention. Fig. 11.4 gives the shock exposure elasticity for the internal specification and Fig. 11.5 the same for the external specification for a growth-rate shock to the investment-capital ratio. In all cases, $\rho = 1$ in order to feature this alternative intertemporal substitution mechanism. We report results for different values of λ and τ. We see in Fig. 11.4 that responses have different signs depending on whether λ is positive or negative. The most pronounced responses are for $\tau = .01$ for which the two different values of λ have offsetting effects that are roughly comparable. In contrast, for the external specification, the $\lambda = -2$ responses are in sharp contrast to those of $\lambda = .67$. The implied intertemporal complementarity in preferences, or so called ``habit persistence'', is particularly prominent when $\tau << 1$.

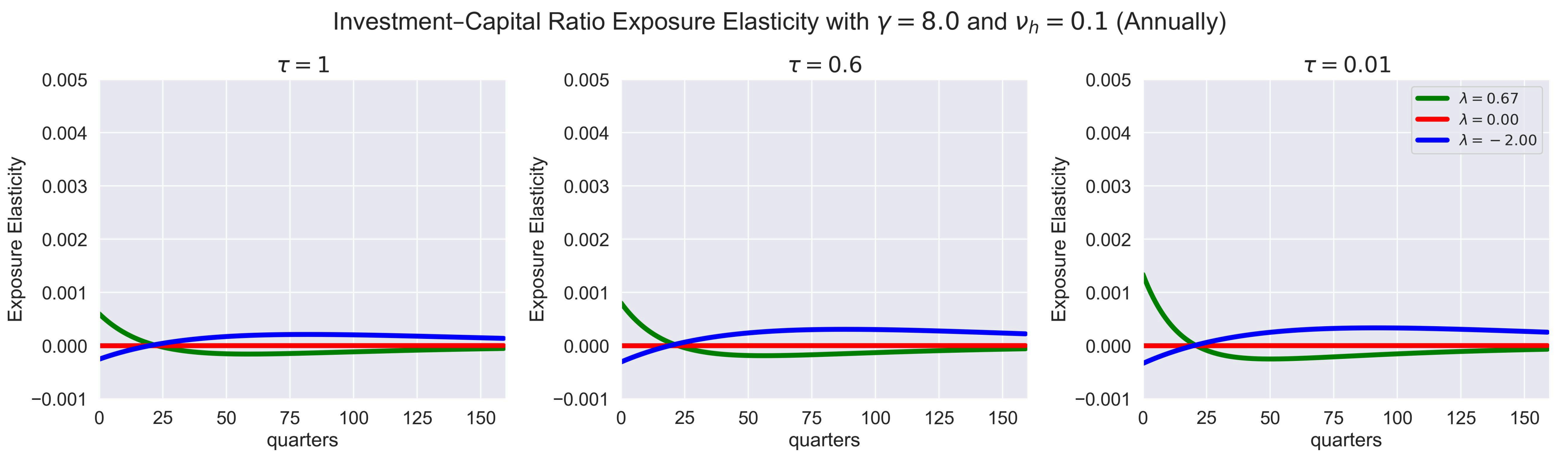

Fig. 11.4 Exposure elasticity of the growth shock in the internal intertemporal preference model for $\lambda = .67, 0, -2$, $\gamma = 8$ and different values of τ.

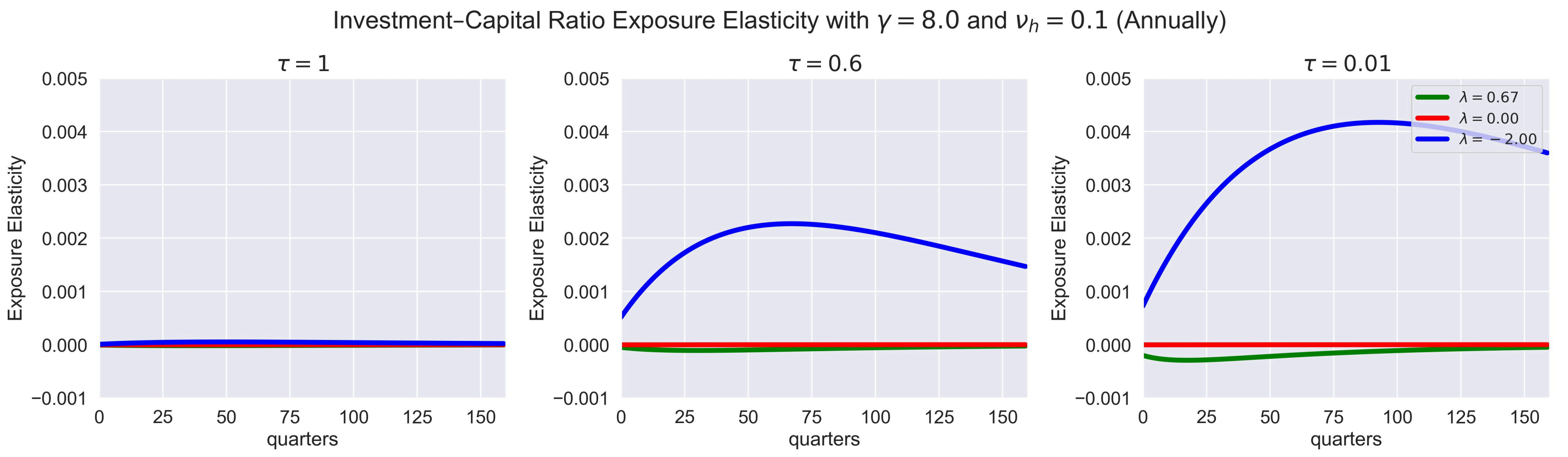

Fig. 11.5 Exposure elasticity of the growth shock in external intertemporal preference model for $\lambda = .67, 0, -2$, $\gamma = 8$ and different values of τ.

Fig. 11.6 and Fig. 11.7 depict the consumption price elasticities for the growth rate shock for both the internal and external specifications when $\gamma = 8$. We again consider intertemporal substitution ($\lambda = .67$) and intertemporal complementarity ($\lambda = -2$). For the internal preference specification the price elasticities are a little bit higher under intertemporal complementarity, but the impact is much more substantial for the external preference specification, particularly when $\tau = .01$.

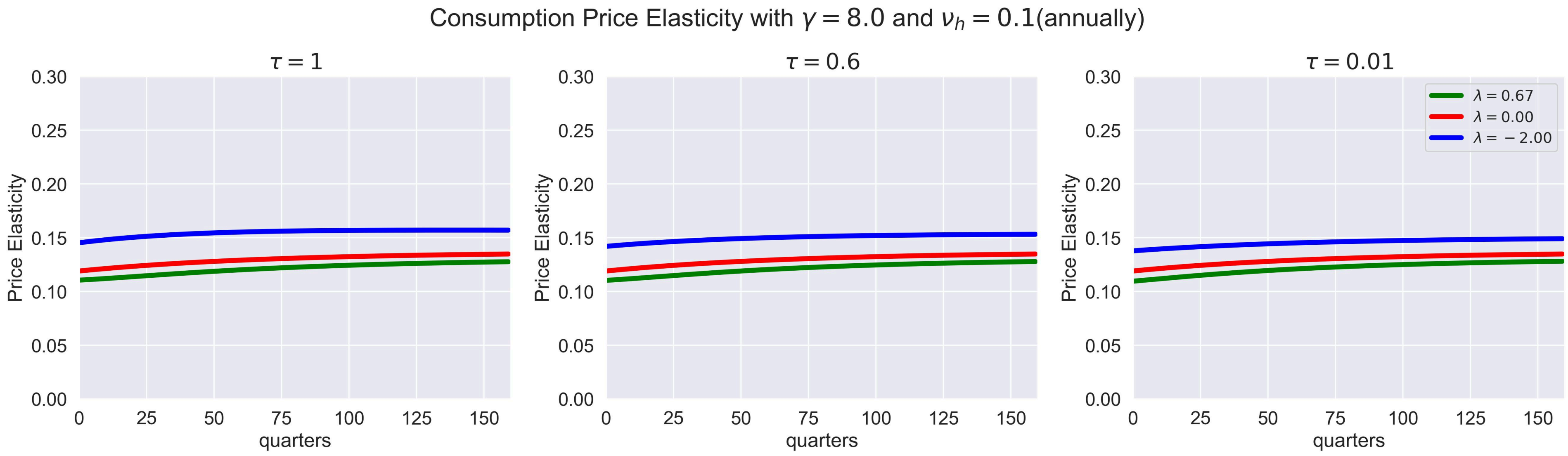

Fig. 11.6 Consumption price elasticity of the growth shock in internal intertemporal preference model for $\lambda = .67, 0, -2$, $\gamma = 8$ and different values of τ.

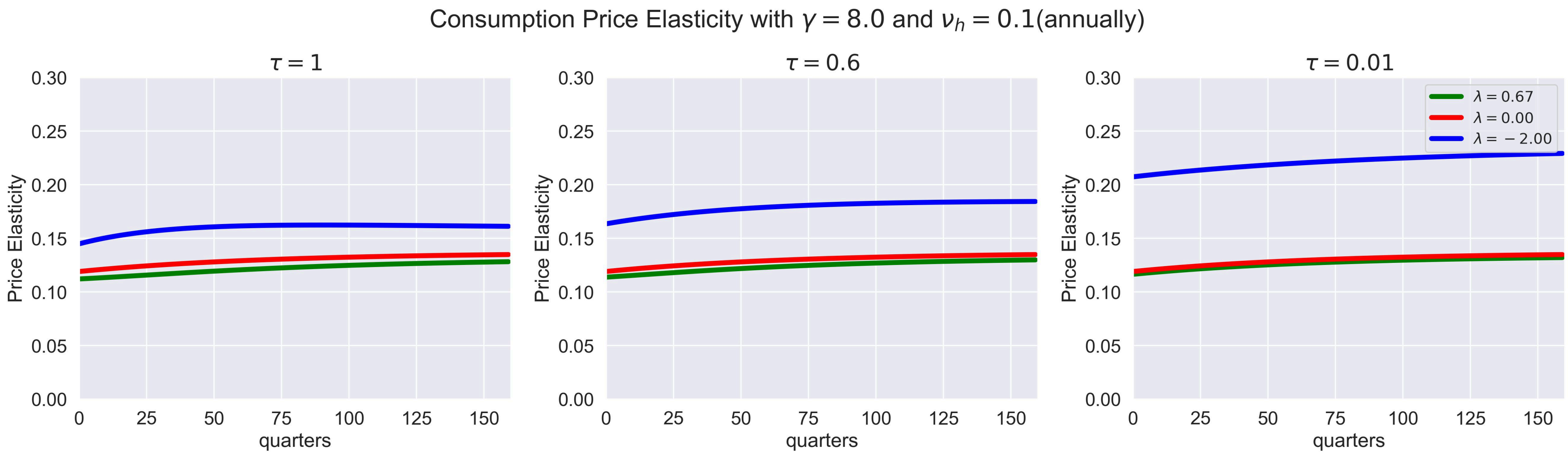

Fig. 11.7 Consumption price elasticity of the growth shock in external intertemporal preference model for $\lambda = .67, 0, -2$, $\gamma = 8$ and different values of τ.

Fig. 11.8 and Fig. 11.9 repeats the plots except lowering $\gamma = 4$. Perhaps not surprisingly, the price elasticities are now half or less what they were when $\gamma = 8$, but the overall patterns remain the same.

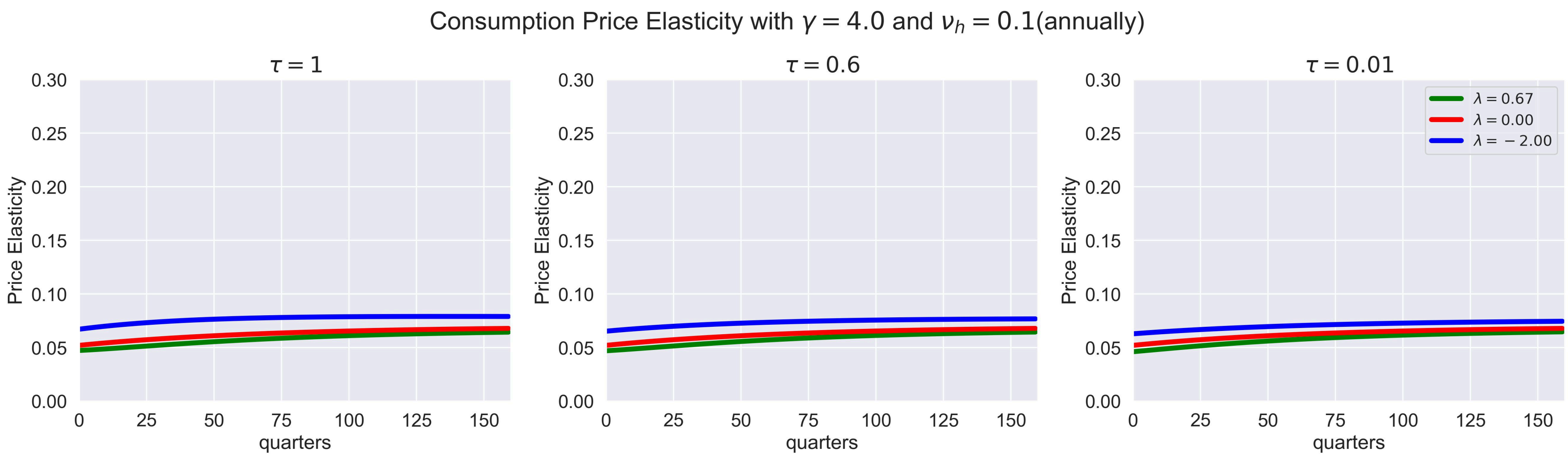

Fig. 11.8 Consumption price elasticity of the growth shock in internal intertemporal preference model for $\lambda = .67, 0, -2$, $\gamma = 4$ and different values of τ.

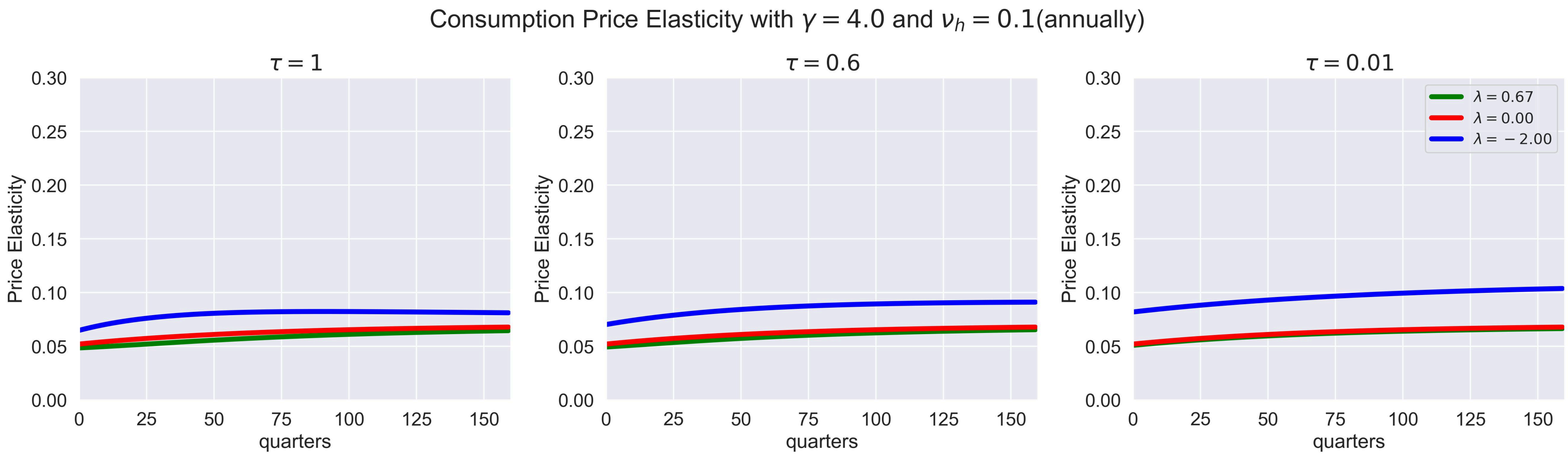

Fig. 11.9 Consumption price elasticity of the growth Shock in external intertemporal preference model for $\lambda = .67, 0, -2$, $\gamma = 4$ and different values of τ.

11.8. Solving models

In this section, we briefly describe one way to extend the approach that builds directly on previous second-order approaches of [Kim *et al.*, 2008], [Schmitt-Grohé and Uribe, 2004], and [Lombardo and Uhlig, 2018]. While such methods should not be viewed as being generically applicable to nonlinear stochastic equilibrium models, we find them useful pedagogically and often at least as initial steps to understanding models that are arguably "smooth." See [Pohl *et al.*, 2018] for a careful study of nonlinearity in asset pricing models with recursive utility.[6]

We implement these methods for second-order approximation using the following steps.

1. Solve for the $\mathsf{q} = 0$ deterministic model.
2. Take as given first and second-order approximate solutions for $\widehat{C}_t - \widehat{G}_t$ and $\widehat{G}_{t+\epsilon} - \widehat{G}_t$. Solve for the approximate solutions for $\widehat{V}_t - \widehat{G}_t$, $\widehat{V}_{t+\epsilon} - \widehat{R}_t$ and $N_{t+\epsilon}$.
3. Compute the first-order expansion and solve the resulting equations following the previous literature for D_t, $\widehat{C}_t - \widehat{G}_t$, and $\widehat{G}_{t+\epsilon} - \widehat{G}_t$. When constructing these equations, use expectations computed using the probabilities induced by $N^0_{t+\epsilon}$. Substitute the first-order approximation for $\widehat{R}_t - \widehat{G}_t$.
4. Compute the second-order expansion and solve the resulting equations following the previous literature. Again use the expectations induced by $N^0_{t+\epsilon}$. In addition, make another recursive utility adjustment expressed in terms of the approximations of $\widehat{R}_t - \widehat{G}_t$.
5. Return to step 2, and repeat until convergence.

Initialize this algorithm by solving the case $\gamma_o = 1$ and $\rho = 1$, which can be computed without iteration.

See the Appendices that follow for more details and formulas to use in the solution method.

As a second approach, we iterate over an $N^*_{t+\epsilon}$ given by formula (11.32) approximation restricted to induce an alternative probability distribution. Call the approximation $\widetilde{N}_{t+\epsilon}$ with an induced distribution for $W_{t+\epsilon}$ that is normal with conditional mean $\tilde{\mu}_t$ and covariance matrix $\widetilde{\Sigma}$.

While we discussed the approximation for resource allocation problems with recursive utility, there is a direct extension of this approach to solve a general class of stochastic equilibrium models by stacking a system of expectational-type equations expressed in part using the recursive utility stochastic discount factor that we derived. For resource allocation problems, we expressed the first-order conditions for the planner in utility units, which simplified some formulas. Equilibrium models not derived from a planner's problems typically use stochastic discount factors expressed in consumption units when representing investment choices. The approximation methods described in this chapter have a direct extension to such models.

11.9. Appendix A: Solving the planner's problem

Write the system of interest, including the state equations (11.33), the consumption equation and static constraint (11.34), the first-order conditions (11.35), and the co-state evolution (11.36) as:

$$\begin{aligned} X_{t+1}(\mathsf{q}) &= \psi^x\left[D_t(\mathsf{q}), X_t(\mathsf{q}), \mathsf{q}W_{t+1}, \mathsf{q}\right], \\ \log G_{t+1}(\mathsf{q}) - \log G_t(\mathsf{q}) &= \psi^g\left[D_t(\mathsf{q}), X_t(\mathsf{q},), \mathsf{q}W_{t+1}, \mathsf{q}\right], \\ \widehat{C}_t(\mathsf{q}) &= \kappa\left[D_t(\mathsf{q}), X_t(\mathsf{q})\right] + \widehat{G}_t(\mathsf{q}), \\ 0 &= \phi[D_t(\mathsf{q}), X_t(\mathsf{q})], \\ Q_t \mathbb{E}\left(N_{t+1} H_{t+1} \mid \mathfrak{A}_t\right) + P_t L_t - M_t &= 0 \end{aligned}$$

where

$$Q_t \stackrel{\text{def}}{=} \beta \exp\left[(1-\rho)\left(\widehat{R}_t - \widehat{V}_t\right)\right]$$

$$P_t \stackrel{\text{def}}{=} (1-\beta)\exp\left[(\rho-1)\left(\widehat{V}_t - \widehat{G}_t\right)\right]$$

$$H_{t+1} \stackrel{\text{def}}{=} \begin{bmatrix} \psi^x_{d'}(D_t, X_t, W_{t+1})' & \psi^g_{d'}(D_t, X_t, W_{t+1}) \\ \psi^x_{x'}(D_t, X_t, W_{t+1})' & \psi^g_{x'}(D_t, X_t, W_{t+1})' \\ \psi^x_{g'}(D_t, X_t, W_{t+1})' & 1 \end{bmatrix} \begin{bmatrix} MX_{t+1} \\ MG_{t+1} \end{bmatrix}$$

$$\begin{aligned} L_t &\stackrel{\text{def}}{=} \left[\exp\left[(1-\rho)\left(\widehat{C}_t - \widehat{G}_t\right)\right] \begin{bmatrix} \kappa_d(D_t, X_t) \\ \kappa_x(D_t, X_t) \\ 1 \end{bmatrix}\right] \\ M_t &\stackrel{\text{def}}{=} \begin{bmatrix} 0 \\ MX_t \\ MG_t \end{bmatrix} + \begin{bmatrix} \phi_{d'}(D_t, X_t)' \\ \phi_{x'}(D_t, X_t)' \\ 0 \end{bmatrix} MS_t. \end{aligned}$$

Here, for computational purposes, we use that $\widehat{C}_t - \widehat{V}_t = \left(\widehat{C}_t - \widehat{G}_t\right) - \left(\widehat{V}_t - \widehat{G}_t\right)$. We solve for $\widehat{C}_t - \widehat{G}_t, D_t, MX_t$ as a function of X_t. The objects: $\widehat{C} - \widehat{G}, D$ and MX are sometimes referred to as jump variables since we do not impose initial conditions for these variables as part of a solution.

Our solution will entail an iteration. We will impose a specification for Q^1, Q^2, and N and find an approximate solution for the dynamical system. Then given this solution, we will compute a new implied solution for Q^1, Q^2, and N. We then iterate this until we achieve numerical convergence. We use second-order approximations for both steps.

11.9.1. Some steady state calculations

Observe from the recursive utility updating that:

$$\widehat{V}_t^0 - \widehat{G}_t^0 = \frac{1}{1-\rho}\log\left[(1-\beta)\exp\left[(1-\rho)\left(\widehat{C}_t^0 - \widehat{G}_t^0\right)\right] + \beta\exp\left[(1-\rho)\left(\widehat{R}_t^0 - \widehat{G}_t^0\right)\right]\right]$$
$$\widehat{R}_t^0 - \widehat{G}_t^0 = \widehat{V}_{t+1}^0 - \widehat{G}_{t+1}^0 + \widehat{G}_{t+1}^0 - \widehat{G}_t^0.$$

In the steady state we view this as two equations in three variables: $\widehat{V}_t^0 - \widehat{G}_t^0, \widehat{R}_t^0 - \widehat{G}_t^0$ and $\widehat{G}_{t+1}^0 - \widehat{G}_t^0$, each of which we assume is time invariant.
We construct

$$\begin{aligned} Q_t^0 =& \beta\exp\left[(1-\rho)\left(\widehat{R}_t^0 - \widehat{V}_t^0\right)\right] \\ =& \beta\exp\left[(1-\rho)\left(\widehat{G}_{t+1}^0 - \widehat{G}_t^0\right)\right], \end{aligned}$$

and

$$P_t^0 = (1-\beta)\exp\left[(\rho-1)\left(\widehat{V}_t^0 - \widehat{G}_t^0\right)\right]$$

in the steady state, and use them to construct the remaining steady-state equations:

$$Q_t^0 H_{t+1}^0 + P_t^0 - M_t^0 = 0.$$

In this equation, H_{t+1}^0, L_t^0, and M_t^0 are constructed from the formulas for H_{t+1}, L_t, and M_t, defined previously, by setting the shock vector to zero and treating the relevant variables as time invariant.

11.9.2. Q and P derivatives

For the order one, write

$$Q_t^1 \overset{\text{def}}{=} (1-\rho)Q_t^0\left(\widehat{R}_t^1 - \widehat{V}_t^1\right)$$

To compute this contribution, we use equation (11.19) to write

$$\widehat{R}_t^1 - \widehat{C}_t^1 = \frac{1}{\lambda}\left(\widehat{V}_t^1 - \widehat{C}_t^1\right).$$

We then construct

$$\begin{aligned}\widehat{R}_t^1 - \widehat{V}_t^1 =& \left(\widehat{R}_t^1 - \widehat{C}_t\right) + \left(\widehat{C}_t^1 - \widehat{V}_t\right) \\ =& \left(\frac{1-\lambda}{\lambda}\right)\left[\left(\widehat{V}_t^1 - \widehat{G}_t\right) - \left(\widehat{C}_t - \widehat{G}_t\right)\right].\end{aligned}$$

To compute $\widehat{V}_t - \widehat{G}_t$, we rewrite (11.24) as:

$$\begin{aligned}&\widehat{V}_t^1 - \widehat{G}_t^1 = (1-\lambda)\left(\widehat{C}_t^1 - \widehat{G}_t^1\right) \\ &+ \left(\frac{\lambda}{1-\gamma_o}\right)\log\mathbb{E}\left(\exp\left[(1-\gamma_o)\left[\left(\widehat{V}_{t+1}^1 - \widehat{G}_{t+1}^1\right) + \left(\widehat{G}_{t+1}^1 - \widehat{G}_t^1\right)\right]\right] \mid \mathfrak{A}_t\right)\end{aligned}$$

and solve this equation forward by first computing the $\gamma_o = 1$ answer and then adjusting this answer for $\gamma_o > 1$ analogous to the approach described in Remark 11.5.

For the order two approximation,

$$Q_t^2 \stackrel{\text{def}}{=} (1-\rho)^2 Q_t^0\left(\widehat{R}_t^1 - \widehat{V}_t^1\right)^2 + (1-\rho)Q_t^0\left(\widehat{R}_t^2 - \widehat{V}_t^2\right).$$

Express:

$$\widehat{R}_t^2 - \widehat{V}_t^2 = \left(\widehat{R}_t^2 - \widehat{G}_t^2\right) - \left(\widehat{V}_t^2 - \widehat{G}_t^2\right).$$

It follows from (11.31) that

$$\begin{aligned}\widehat{V}_t^2 - \widehat{G}_t^2 = \lambda\mathbb{E}&\left(N_{t+1}^0\left[\left(\widehat{V}_{t+1}^2 - \widehat{G}_{t+1}^2\right) + \left(\widehat{G}_{t+1}^2 - \widehat{G}_t^2\right)\right] \mid \mathfrak{A}_t\right) \\ &+ (1-\lambda)\left(\widehat{C}_t^2 - \widehat{G}_t^2\right) + (1-\rho)(1-\lambda)\lambda\left(\widehat{R}_t^1 - \widehat{G}_t^1 + \widehat{G}_t^1 - \widehat{C}_t\right)^2,\end{aligned}$$

which we solve this equation forward under the N^0 implied change in probability measure. Form:

$$\begin{aligned}\widehat{R}_t^2 - \widehat{G}_t^2 =& \left(\frac{1}{\lambda}\right)\left(\widehat{V}_t^2 - \widehat{G}_t^2\right) - \left(\frac{1-\lambda}{\lambda}\right)\left(\widehat{C}_t^2 - \widehat{G}_t^2\right) \\ &- (1-\rho)(1-\lambda)\left(\widehat{R}_t^1 - \widehat{G}_t^1 + \widehat{G}_t^1 - \widehat{C}_t\right)^2.\end{aligned}$$

The P approximations use some of these same computations where

$$
\begin{aligned}
P_t^0 &\overset{\text{def}}{=} (1-\beta)\exp\left[(\rho-1)\left(\widehat{V}_t^0 - \widehat{G}_t^0\right)\right] \\
P_t^1 &\overset{\text{def}}{=} (\rho-1)P_t^0\left(\widehat{V}_t^1 - \widehat{G}_t^1\right) \\
P_t^2 &\overset{\text{def}}{=} (\rho-1)^2 P_t^0\left(\widehat{V}_t^1 - \widehat{G}_t^1\right)^2 + (1-\rho)P_t^0\left(\widehat{V}_t^2 - \widehat{G}_t^2\right).
\end{aligned}
$$

11.9.3. N derivatives

$$
N_{t+1}^0 \overset{\text{def}}{=} \exp\left[(1-\gamma_o)\left(\widetilde{V}_{t+1}^0 - \widetilde{R}_t^0\right)\right] = \exp\left[(1-\gamma_o)\left(\widehat{V}_{t+1}^1 - \widehat{R}_t^1\right)\right]
$$

$$
\begin{aligned}
N_{t+1}^1 &\overset{\text{def}}{=} \frac{d}{d\mathsf{q}}\exp\left[(1-\gamma_o)\left(\widetilde{V}_{t+1} - \widetilde{R}_t\right)\right]\Big|_{\mathsf{q}=0} \\
&= (1-\gamma_o)N_{t+1}^0\left(\widetilde{V}_{t+1}^1 - \widetilde{R}_t^1\right) \\
&= \left(\frac{1-\gamma_o}{2}\right)N_{t+1}^0\left(\widehat{V}_{t+1}^2 - \widehat{R}_t^2\right).
\end{aligned}
$$

Form:

$$
\begin{aligned}
\widehat{V}_{t+1}^1 - \widehat{R}_t^1 &= \left(\widehat{V}_{t+1}^1 - \widehat{G}_{t+1}^1\right) + \left(\widehat{G}_{t+1}^1 - \widehat{G}_t^1\right) - \left(\widehat{R}_t^1 - \widehat{G}_t^1\right) \\
\widehat{V}_{t+1}^2 - \widehat{R}_t^2 &= \left(\widehat{V}_{t+1}^2 - \widehat{G}_{t+1}^2\right) + \left(\widehat{G}_{t+1}^2 - \widehat{G}_t^2\right) - \left(\widehat{R}_t^2 - \widehat{G}_t^2\right).
\end{aligned}
$$

It can be verified that both N_{t+1}^1 and N_{t+1}^2 have conditional expectations equal to zero. Express

$$
\begin{aligned}
\widehat{V}_{t+1}^1 - \widehat{R}_t^1 &= \left(\frac{1}{1-\gamma_o}\right)\left[\mu^0\cdot(W_{t+1}-\mu^0) + \frac{1}{2}\mu^0\cdot\mu^0\right] \\
\widehat{V}_{t+1}^2 - \widehat{R}_t^2 &= \frac{1}{2}\left(W_{t+1}-\mu^0\right)'\Upsilon_2^2\left(W_{t+1}-\mu^0\right) - \frac{1}{2}\operatorname{tr}\left(\Upsilon_2^2\right) \\
&\quad - \left(W_{t+1}-\mu^0\right)'\left(\Upsilon_1^2 X_t^1 + \Upsilon_0^2\right).
\end{aligned}
\tag{11.42}
$$

In producing these representations, we use that have conditional $\widehat{V}_{t+1}^2 - \widehat{R}_t^2$ and $\widehat{V}_{t+1}^2 - \widehat{R}_t^2$ have mean zero under the conditional probability distribution induced by N_{t+1}^0.

11.10. Appendix B: Approximation formulas (approach one)

Consider the equation:

$$Q_t \mathbb{E}\left(N_{t+1} H_{t+1} \mid \mathfrak{A}_t\right) + P_t L_t - M_t = 0.$$

not including the state evolution equations.

11.10.1. Order zero

The order zero approximation of the product: $N_{t+1} Q_t H_{t+1} + P_t L_t - M_0$ is:

$$Q_t^0 N_{t+1}^0 H_{t+1}^0 + P_t^0 L_t^0 - M_0 = 0$$

Thus the order zero approximate equation is:

$$Q_t^0 \mathbb{E}\left[N_{t+1}^0 \left(H_{t+1}^0\right) \mid \mathfrak{A}_t\right] + L_{t+1}^0 = Q_t^0 H_{t+1}^0 + P_t^0 L_t^0 - M_0 = 0$$

since N_{t+1}^0 has conditional expectation equal to one. We add to this subsystem the $\mathbf{q} = 0$ state dynamic equation inclusive of jump variables, and we compute a stable steady state solution.

11.10.2. Order one

The order one approximation of the product: $Q_t N_{t+1} H_{t+1} + P_t L_t - M_t$ is:

$$Q_t^1 N_{t+1}^0 H_{t+1}^0 + Q_t^0 N_{t+1}^1 H_{t+1}^0 + Q_t^0 N_{t+1}^0 H_{t+1}^1 + P_t^0 L_t^1 + P_t^1 L_t^0 - M_t^1.$$

Thus the order one approximate equation is:

$$\begin{aligned} & Q_t^0 \mathbb{E}\left(N_{t+1}^1 H_{t+1}^0 + N_{t+1}^0 H_{t+1}^1 \mid \mathfrak{A}_t\right) + Q_t^1 H_{t+1}^0 + P_t^0 L_t^1 + P_t^1 L_t^0 \\ & = Q_t^0 \mathbb{E}\left(N_{t+1}^0 H_{t+1}^1 \mid \mathfrak{A}_t\right) + Q_t^1 H_{t+1}^0 + P_t^0 L_t^1 + P_t^1 L_t^0 - M_t^1 \\ & = 0 \end{aligned}$$

where we used the implication that $\mathbb{E}\left(N_{t+1}^1 \mid \mathfrak{A}_t\right) = 0.$

11.10.3. Order two

The order two approximation of the product: $Q_t N_{t+1} H_{t+1} + P_t L_t - M_t$ is:

$$\begin{aligned} & Q_t^0 N_{t+1}^0 H_{t+1}^2 + Q_t^0 N_{t+1}^2 H_{t+1}^0 + 2 Q_t^0 N_{t+1}^1 H_{t+1}^1 \\ & + P_t^0 L_t^2 + 2 P_t^1 L_t^1 + P_t^2 L_t^0 - M_t^2 \\ & + 2 N_{t+1}^1 Q_t^1 H_{t+1}^0 + 2 N_{t+1}^0 Q_t^1 H_{t+1}^1 + N_{t+1}^0 Q_t^2 H_{t+1}^0 \end{aligned}$$

The terms $Q_t^0 N_{t+1}^2 H_{t+1}^0$ and $2N_{t+1}^1 Q_t^1 H_{t+1}^0$ have conditional expectation equal to zero. Thus the approximating equation is:

$$\begin{aligned}&Q_t^0 \mathbb{E}\left(N_{t+1}^0 H_{t+1}^2 \mid \mathfrak{A}_t\right) + P_t^0 L_t^2 + 2P_t^1 L_t^1 + P_t^2 L_t^0 \\ &+ 2Q_t^0 \mathbb{E}\left(N_{t+1}^1 H_{t+1}^1 \mid \mathfrak{A}_t\right) + 2Q_t^1 \mathbb{E}\left(N_{t+1}^0 H_{t+1}^1 \mid \mathfrak{A}_t\right) + H_{t+1}^0 Q_t^2 \mathbb{E}\left(N_{t+\epsilon}^0 \mid \mathfrak{A}_t\right) \\ &= 0.\end{aligned}$$

To elaborate on the contributions in the second line, express H_{t+1}^1 as

$$H_{t+1}^1 = \Theta_0^1 + \Theta_1^1 X_t^1 + \Theta_2^1 \left(W_{t+1} - \mu^0\right). \tag{11.43}$$

Then

$$\begin{aligned}2Q_t^0 \mathbb{E}\left(N_{t+1}^1 H_{t+1}^1 \mid \mathfrak{A}_t\right) &= (1-\gamma_o) Q_t^0 \mathbb{E}\left[N_{t+1}^0 \left(\widehat{V}_{t+1}^2 - \widehat{R}_t^2\right) H_{t+1}^1 \mid \mathfrak{A}_t\right], \\ &= (\gamma_o - 1) Q_t^o \Theta_2^1 \left[\Upsilon_1^2 X_t^1 + \Upsilon_0^2\right] \\ 2Q_t^1 \mathbb{E}\left(N_{t+1}^0 H_{t+1}^1 \mid \mathfrak{A}_t\right) &= 2Q_t^0 (1-\rho)\left(\widehat{R}_t^1 - \widehat{V}_t^1\right)\left(\Theta_0^1 + \Theta_1^1 X_t^1\right), \\ H_{t+1}^0 Q_t^2 \mathbb{E}\left(N_{t+1}^0 \mid \mathfrak{A}_t\right) &= H_{t+1}^0 Q_t^0 \left[(1-\rho)^2 \left(\widehat{R}_t^1 - \widehat{V}_t^1\right)^2 + (1-\rho)\left(\widehat{R}_t^2 - \widehat{V}_t^2\right)\right].\end{aligned}$$

The formula for the first of these terms follows from (11.42) and (11.43), along with the fact that the third central moments of normals are zero.

We add to this second-order subsystem, the second-order approximation of the state dynamics inclusive of the jump variables. We substitute in the solution for the first-order approximation for the jump variables into both the first and second-order approximate state dynamics. In solving the second-order jump variable adjustment we use expectations induced by N_{t+1}^0 throughout under which W_{t+1} is conditionally normally distributed with mean μ^0 and covariance I.

11.11. Appendix C: Approximation formulas (approach two)

In this approach we use the same order zero approximation. For the order one approximation, we use the formula (11.32) for $\widetilde{N}_{t+1}$, which approximates N_{t+1}^*, in conjunction with:

$$Q_t^0 \mathbb{E}\left(\widetilde{N}_{t+1} H_{t+1}^1 \mid \mathfrak{A}_t\right) + Q_t^1 H_{t+1}^0 + P_t^0 L_t^1 + P_t^1 L_t^0 - M_t^1 = 0. \tag{11.44}$$

From formula (11.42), it follows that under the $\widetilde{N}_{t+1}$, induced change in probability, W_{t+1} is normally distributed with conditional mean

$$\mu^0 + (1-\gamma_o)\left(\Upsilon_2^2\right)^{-1}\left(\Upsilon_1^2 X_t^1 + \Upsilon_0^2\right),$$

and conditional precision:

$$(\gamma_o - 1)\Upsilon_2^2 + \mathbb{I}.$$

For the order two approximation, we use:

$$Q_t^0 \mathbb{E}\left(\widetilde{N}_{t+1} H_{t+1}^2 \mid \mathfrak{A}_t\right) + P_t^0 L_t^2 + 2P_t^1 L_t^1 + P_t^2 L_t^0 - M_t^2 + 2Q_t^1 \mathbb{E}\left(\widetilde{N}_{t+1} H_{t+1}^1 \mid \mathfrak{A}_t\right) + H_{t+1}^0 Q_t^2 = 0.$$

11.12. Appendix D: Parameter values

To facilitate a comparison to a global solution method, we write down a discrete-time approximation to a continuous time version of such an economy. (See Section 4.4 of [Hansen *et al.*, 2024] for a continuous-time benchmark model that our discrete-time system approximates. Note that we convert the annual parameters in that paper to quarterly time.) The parameter settings are:

δ	ι_k	ζ	ν_k	ν_1	ν_2	μ_2
0.025	0.01	32	0.01	0.014	0.0485	6.3×10^{-6}

$$\begin{bmatrix}\sigma_k \\ \sigma_1 \\ \sigma_2\end{bmatrix} = \sqrt{3}\begin{bmatrix} .92 & .40 & 0 \\ 0 & 5.7 & 0 \\ 0 & 0 & .00031\end{bmatrix}$$

The numbers for $\eta_k, \phi, \beta_1, \sigma_k$ and σ_1 are such that, when multiplied by stochastic volatility, they match the parameters from [Hansen and Sargent, 2021]. In particular, the constant Z^2, which scales our σ_k to match is 7.6×10^{-6}, which is the 67th percentile of our Z^2 distribution. While [Hansen and Sargent, 2021] use a lower triangular representation for the two-by-two right block of

$$\begin{bmatrix}\sigma_k \\ \sigma_1\end{bmatrix},$$

we use an observationally equivalent upper triangular representation for most of the results. Finally, the numbers for β_2 and σ_2 come from [Schorfheide *et al.*, 2018], but they are adjusted for approximation purposes as described in Appendix A [Hansen *et al.*, 2024]. In both cases, we use the medians of their econometric evidence as input into our analysis.

For the extension to the habit persistence model in Section 10.7.5, we use a habit persistence of $\nu_h = 0.025$.

[1] In general, this exponential martingale formula produces a local martingale with conditional expectations that might decline over time. There are a variety of sufficient conditions that can be checked to verify that the constructed process is actually a martingale with unit expectation.

[2] [Lombardo and Uhlig, 2018] provides a discussion of how their approach builds on more general perturbation methods as discussed by [Holmes, 2012] and [Judd, 1998].

[3] See, for instance, [Schmitt-Grohé and Uribe, 2004].

[4] Consistent with overall message of the paper, the [Hansen *et al.*, 2008] predictability evidence turned out to be "fragile" and was modified and updated in [Hansen and Sargent, 2021] Appendix B. This same appendix suggests a way to deduce a statistical approximation to the first order dynamics of [Bansal and Yaron, 2004] from a more general VAR representation of the consumption dynamics.

[5] We normalized the stochastic volatility shock σ_x^2 to be negative implying that a positive shock reduces the stochastic volatility state variable. Under this normalization, the shock price elasticities are positive.

[6] [Pohl *et al.*, 2018] provide examples of when log-linear or local methods of computation fail to provide good approximations.