Stochastic Responses

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Date: September 2024

10.1. Introduction

Impulse-response methods have been used by economists since [Frisch, 1933] and in other disciplines. For nonlinear stochastic models, impulse responses are themselves stochastic. Alternative approaches have been suggested in economics including [Gallant *et al.*, 1993], [Koop *et al.*, 1996], and [Gourieroux and Jasiak, 2005]. This chapter provides stochastic responses in both discrete and continuous time for marginal changes in state variables and shocks. The vast literature on vector autoregressions views these responses as ends in and of themselves. Identified shocks as exogenous inputs into a dynamical system in effect 'cause' movements in the vector time series of interest. Such measurements, while interesting, can have rather indirect connections to hypothetical interventions related to perspective policy changes that are at the heart of structural models. Builders of dynamic stochastic equilibrium models use the construct of 'structural' in the sense of [Marschak, 1953], [Hurwicz, 1966], and [Lucas, 1976]. They allow for investigating how a dynamical system changes when one portion of it is altered. For us, these stochastic responses are central inputs into marginal valuations that we will use for a variety of purposes. In subsequent chapters, we provide extensive discussions of two types of such applications. The first type provides asset-pricing type representations for endogenous variables including various forms of capital. These include policy relevant variables such as the social cost of climate change and the social value of research and development. The second type generates what we call shock elasticities that help us characterize the building blocks for exposures to uncertainty and prices of those exposures.

10.2. Discrete time

We first consider a discrete-time specification.

10.2.1. Discrete-time Markov dynamics

We start with Markov process

$$\begin{aligned} X_{t+1} &= \psi(X_t, W_{t+1}), \\ Y_{t+1} - Y_t &= \kappa(X_t, W_{t+1}), \end{aligned} \tag{10.1}$$

where there are n components of X, Y is scalar, and W is k dimensional.

10.2.2. Discrete-time variational dynamics

Let Λ denote the first variational process for X, and let Δ denote the first variational process for Y. These variational processes are the ingredients to stochastic impulse responses to small changes in the underlying state variables. We compute them by "differentiating" in a generalized sense that accommodates the underling stochastic structure. To obtain a recursive representation for (Λ, Δ), we differentiate (10.1) and apply the chain rule:

$$\begin{aligned} \Lambda_{t+1} &= \frac{\partial \psi}{\partial x'}(X_t, W_{t+1})\Lambda_t \\ \Delta_{t+1} - \Delta_t &= \frac{\partial \kappa}{\partial x}(X_t, W_{t+1}) \cdot \Lambda_t. \end{aligned} \tag{10.2}$$

In this calculation, Λ_{t+1} and Δ_{t+1} are stochastic as they inherit the stochastic dependence of X_{t+1} and Y_{t+1}. By differentiating the process at a given calendar date, we are allowing for date t variables to change as a function of date t information.

To obtain alternative stochastic (local) impulse response functions, we initialize $({\Lambda_0}', \Delta_0)'$ to be one of the coordinate vectors that depends on one of the initial states that we want to perturb. Then $({\Lambda_t}', \Delta_t)'$ is the date t state vector stochastic response to the perturbation of the initial value of the component.

To perturb Y_0, we can set $\Lambda_0 = 0$ and $\Delta_0 = 1$; then $\Lambda_t = 0$ and $\Delta_t = 1$ for all t. Alternatively, if we initialize Λ_0 be a coordinate vector and $\Delta_0 = 0$, then the response Δ_t will be a stochastic process. The outcome of the coordinate vector initialization is a stochastic local impulse response to a marginal change in a particular state variable.

Remark 10.1

The evolution of the variational processes is nonstochastic if $\frac{\partial \psi}{\partial x'}$ and $\frac{\partial \kappa}{\partial x}$ are constant as is true when ψ and κ are affine in x. Otherwise, variational processes are stochastic.

Example 10.1

Consider the following quadratic specification:

$$\begin{aligned} X_{t+1}^i &= \mathsf{a}_i \cdot X_t + \frac{1}{2} X_t{}' \mathbb{A}_i X_t + X_t{}' \mathbb{B}_i W_{t+1} + \mathsf{b}_i \cdot W_{t+1}, \quad i = 1, \ldots, n \\ Y_{t+1} - Y_t &= \mathsf{d} \cdot X_t + \frac{1}{2} X_t{}' \mathbb{D} X_t + X_t{}' \mathbb{F} W_{t+1} + \mathsf{f} \cdot W_{t+1}. \end{aligned}$$

where $\mathbb{A}_i$ and $\mathbb{D}$ are symmetric. A simple calculation shows:

$$\begin{aligned} \Lambda_{t+1}^i &= \mathsf{a}_i \cdot \Lambda_t + \Lambda_t{}' \mathbb{A}_i X_t + \Lambda_t{}' \mathbb{B}_i W_{t+1}, \quad i = 1, \ldots, n \\ \Delta_{t+1} - \Delta_t &= \mathsf{d} \cdot \Lambda_t + \Lambda_t{}' \mathbb{D} X_t + \Lambda_t{}' \mathbb{F} W_{t+1}. \end{aligned}$$

10.3. Continuous-time dynamics

We now consider the continuous-time counterpart for Brownian motion shocks.

10.3.1. Markov diffusion dynamics

As a part of a more general derivation, we begin with state dynamics modeled as a Markov diffusion:

$$\begin{aligned} dX_t &= \mu(X_t) dt + \sigma(X_t) dW_t \\ dY_t &= \nu(X_t) dt + \varsigma(X_t) \cdot dW_t. \end{aligned}$$

where W is now a k-dimensional standard Brownian motion. We denote the filtration (family of specifications of conditioning information events) $\mathfrak{F} \overset{\text{def}}{=} \{\mathfrak{F}_t : t \geq 0\}$ constructed from the Brownian motion and any pertinent date zero information.

10.3.2. Variational process

Following [Borovička *et al.*, 2014], we construct marginal impulse response functions using what are called variational processes. We build the dynamics for what is called the first variational, Λ by following the construction in [Fournie *et al.*, 1999]. The first variational process tells the marginal impact on future X of a marginal change in one of the initial states analogous to the Λ process that we constructed in discrete time. Thus this process has the same number of components as X. By initializing the process at one of the alternative coordinate vectors, we again isolate an initial state of interest.[1].

The drift for the i^{th} component of Λ is

$$\lambda' \frac{\partial \mu_i}{\partial x}(x)$$

and the coefficient on the Brownian increment is

$$\lambda' \frac{\partial \sigma_i}{\partial x}(x)$$

for λ a hypothetical realization of Λ_t and x a hypothetical realization of X_t, where $'$ denotes vector or matrix transposition. The implied evolution of the process Λ^i is[2]

$$d\Lambda_t^i = (\Lambda_t)' \frac{\partial \mu_i}{\partial x}(X_t)dt + (\Lambda_t)' \frac{\partial \sigma_i}{\partial x}(X_t)dW_t.$$

With the appropriate stacking, the drift for the composite process (X, Λ) is:

$$\mu^a(x, \lambda) \overset{\text{def}}{=} \begin{bmatrix} \mu(x) \\ \lambda' \frac{\partial \mu_i}{\partial x}(x) \\ \cdots \\ \lambda' \frac{\partial \mu_n}{\partial x}(x) \end{bmatrix}, \tag{10.3}$$

and the composite matrix coefficient on dW_t is given by

$$\sigma^a(x, \lambda) \overset{\text{def}}{=} \begin{bmatrix} \sigma(x) \\ \lambda' \frac{\partial \sigma_1}{\partial x}(x) \\ \cdots \\ \lambda' \frac{\partial \sigma_n}{\partial x}(x) \end{bmatrix}. \tag{10.4}$$

Let Δ be the scalar variational process associated with Y. Then

$$d\Delta_t = \Lambda_t \cdot \frac{\partial \nu}{\partial x}(X_t)dt + {\Lambda_t}' \frac{\partial \varsigma}{\partial x'} dW_t$$

Analogous to the discrete-time outcome, the variational dynamics depend explicitly on the original diffusion dynamics. As in discrete time, by initializing the vector Λ_0 at a coordinate vector, the resulting processes give marginal responses to a corresponding state vector.

Example 10.2

Consider the case of linear dynamics:

$$\begin{matrix} \mu(x) = \mathbb{A}x & \sigma(x) = \mathbb{B} \\ \nu(x) = \mathbb{D}x & \varsigma(x) = \mathbb{F} \end{matrix}.$$

Then

$$\mu^a(x, \lambda) = \begin{bmatrix} \mathbb{A}x \\ \mathbb{A}\lambda \end{bmatrix}$$

$$\sigma^a(X, \lambda) = \begin{bmatrix} \mathbb{B} \\ 0 \end{bmatrix}.$$

Thus

$$\Lambda_t = \exp{(\mathbb{A}t)}\Lambda_0,$$

and

$$\begin{aligned} \Delta_t &= \int_0^t \mathbb{D}\Lambda_u du + \Delta_0 \\ &= \left[\int_0^t \mathbb{D}\exp{(\mathbb{A}u)}du\right]\Lambda_0 + \Delta_0 \\ &= -\mathbb{A}^{-1}\left[\mathbb{I} - \exp{(\mathbb{A}t)}\right]\Lambda_0 + \Delta_0. \end{aligned}$$

Given the underlying linearity, the local responses coincide with global responses.

In the calculations that follow, let Λ^j be the variational process for which Λ_0^j is a coordinate vector with a one in position j. From the composite processes

$$\Lambda^a = [\Lambda_1 \quad \ldots \quad \Lambda_n].$$

Also we will have cause to do a forward shift $\mathbb{S}^\tau$ of these process by which we shift the time units on all of the variables used in the the construction and the initialization period forward τ time periods.

10.3.3. Responses to initial shocks

So far, we have characterized stochastic responses to initial changes in the state variables. From these, we deduce vector of responses to the initial shocks:

$$\Phi = \Lambda^a \sigma(X_0) \quad \Psi = \Lambda^a \sigma(X_0) + \varsigma(X_0)$$

Under nonlinearity, these responses will be stochastic just as with the state variable perturbations. For the special case of linear dynamics given in Example ,

$$\begin{aligned}\Phi_t &= \exp(\mathbb{A}t)\mathbb{B} \\ \Delta_t &= -\mathbb{A}^{-1}\left[\mathbb{I} - \exp(\mathbb{A}t)\right]\mathbb{B} + \mathbb{F},\end{aligned}$$

which are the continuous-time counterparts of the familiar impulse responses.

With Markov diffusions, we also have a state-dependent counterpart to a moving-average representation that is well known from linear time series models. The resulting formula is known as the Haussmann-Clark-Ocone representation and is given by

$$\begin{aligned}X_t &= \int_0^t \mathbb{E}\left(\mathbb{S}^u \Phi_{t-u} \mid \mathfrak{F}_u\right) dW_u^j + \mathbb{E}\left(X_t \mid \mathfrak{F}_0\right) \\ Y_t &= \int_0^t \mathbb{E}\left(\mathbb{S}^u \Psi_{t-u} \mid \mathfrak{F}_u\right) \cdot dW_u^j + \mathbb{E}\left(Y_t \mid \mathfrak{F}_0\right).\end{aligned}$$

Note that we form conditional expectations of time shifted stochastic responses to form the random coefficients in the moving-average representations as given by $\mathbb{E}\left(\mathbb{S}^u \Phi_{t-u} \mid \mathfrak{F}_u\right)$ and $\mathbb{E}\left(\mathbb{S}^u \Psi_{t-u} \mid \mathfrak{F}_u\right)$. When the responses turn out not to be stochastic, as in the case of the Remark 2 example, the conditional expectations and the shift are inconsequential. In this case, we recover the familiar convolution formulas for moving-average representations.

Remark 10.2

Many empirical researchers estimate directly what macroeconomists call Jorda projections. These are implemented by regressing a forward sequence of a scalar process on current variable and a shock or particular interest. One can interpret the ambition as wanting to infer impulse responses from direct regressions of future variables on the initial ones. One can view the ambition as a way to measure impulse responses. For instance, the aim could be to infer:

$$\mathbb{E}\left(\Phi_t \mid \mathcal{F}_0\right) \text{ and } \mathbb{E}\left(\Psi_t \mid \mathcal{F}_0\right), \quad t \geq 0$$

by regressing X_t and Y_t on a measured shock of interest and including additional variables to purge some of the variation in the measured shock. Many applied papers will include cross terms that are pre-determined in advance of the shock to accommodate a form of nonlinearity. For this to be coherent, as our analysis makes clear, one has to think through how the nonlinearity compounds within the stochastic system. The shock of interest can alter other variables that in turn influence the variable of interest in future time periods. Our use of variational processes captures this perspective when the ambition is to measure local impacts.

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[1] Our initial condition for Λ_0 differs from [Fournie *et al.*, 1999] in a superficial way. They treat Λ as a matrix with an identity as the initialization. In this way, they consider all of the states of interest simultaneously. We take Λ to be a vector and characterize the marginal initial responses one at a time by letting the initial condition be any one of the coordinate vectors.

[2] Since we are working with an instantaneous evolution with Brownian increments, we are implicitly appealing to a formalism known as Malliavin calculus.