Homework for Econ 312 Chapter 4

May 5, 2026

1. Suppose that

$$X_t = \sum_{j=0}^{\infty} \beta^j \sigma \cdot W_{t-j}$$

where $\mid \beta \mid < 1$. Also suppose that

$$Y_{t+1} = Y_t + X_{t+1}.$$

(a) Find the martingale component for the Y process and compute its variance of its increment.

(b) Compute the variance of X_t.

(c) Suppose that $\beta > 0$. Which variance is larger, the variance of the martingale increment or the variance of X_t?

(d) Suppose that $\beta < 0$. Now, which variance is larger?

For the last two parts you can establish inequalities formally, or you can i) note that the ratio of the two variances does not depend on σ, and plot the ratio of the two variances as a function of β.

2. (Higher-order VAR represented as a first-order VAR) Suppose that $\{X_t : t = 0, 1, ...\}$ is an n-dimensional stochastic process. Suppose that

$$X_{t+1} = \mathbb{A}_{11} X_t + \mathbb{A}_{12} X_{t-1} + \mathbb{B}_1 W_{t+1}$$

where $\mathbb{B}_1$ is a square matrix and W_{t+1} is normally distributed with mean zero and covariance matrix an n-dimensional identity matrix. Construct

$$X^a_{t+1} = \begin{bmatrix} X_{t+1} \\ X_t \end{bmatrix}.$$

(a) Show that

$$X^a_{t+1}$$

can be represented as a first-order vector autoregression:

$$X^a_{t+1} = \mathbb{A} X^a_t + \mathbb{B} W_{t+1}$$

for $t = 1, 2, ...$ In particular, show how to construct the matrices $\mathbb{A}$ and $\mathbb{B}$.

(b) Suppose that $\mathbb{A}$ is a stable matrix (all of the eigenvalues have absolute values that are less than one). Propose a stationary distribution for $\{X^a_t : t = 1, 2, ...\}$.

This is an illustration of a more general construction for representing a higher-order Markov process as a first-order process.

3. Consider a vector time series model constructed so that

$$X_{t+1} = \mathbb{A} X_t + (\mathbb{I} - \mathbb{A})\, \mu + \mathbb{B} W_{t+1}$$

where $\mathbb{A}$ is a stable matrix,

$$X_t = \begin{bmatrix} Y^1_t - Y^2_t \\ Y^2_t - Y^2_{t-1} \\ X^3_t \end{bmatrix}.$$

(a) Verify that μ is the stationary mean of X_t.

(b) Deduce formulas for κ_1 and κ_2 such that

$$\begin{aligned} Y^1_{t+1} - Y^1_t &= \kappa_1(X_t, W_{t+1}) \\ Y^2_{t+1} - Y^2_t &= \kappa_2(X_t, W_{t+1}) \end{aligned}$$

(c) Deduce the trend-martingale decomposition for the Y^1 and Y^2 processes.

(d) Verify that Y^1 and Y^2 are cointegrated. What is the cointegrating vector?

4. In QuantMFR, Chapter 4.3.2, a (recursive) preference specification is posed and a formula for continuation value $\widehat{V}_t$ for a first-order VAR specification. The remark suggests a way to go from the value function the implied one-period marginal valuations.

(a) Use the formula for the continuation-value process to the one-period "stochastic discount factor" to produce a formula for

$$\log S_{t+1} - \log S_t$$

(b) Suppose that

$$Z_{t+1} = \exp\left(\zeta W_{t+1} - \frac{1}{2}|\zeta|^2\right)$$

where ζ is a row vector. Use properties of the log-normal distribution to show that Z_{t+1} has conditional expectation equal to one.

(c) Produce a formula for its shadow price:

$$\frac{d\pi_t}{d\epsilon}|_{\epsilon=0}.$$

Think of ζ capturing a particular exposure to uncertainty induced by W_{t+1}. Use your answer to discuss how

$$\log \frac{d\pi_t}{d\epsilon}|_{\epsilon=0}.$$

depends on γ.

5. In QuantMFR, Chapter 4.3.4 a model of growth-rate regimes is presented and analyzed. Using the preference specification given previously in Chapter 4.3.2, show how to compute the continuation value process $\{\widehat{V}_t\}$.