Online Appendix to Comparative Valuation Dynamics in Production Economies:

Long-run Uncertainty, Heterogeneity, and Market Frictions

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B Calculations for heterogeneous-agent model

We develop the theoretical model calculations used in the model with agent heterogeneity.

B.1 Setup and HJB equations

For each agent type with discount rate δ, inverse EIS ρ, and risk aversion γ, their HJB equation is given by

$$\begin{aligned} 0 = \max \left(\frac{\delta}{1-\rho}\right)\left[\left(\frac{c}{\exp(\upsilon)}\right)^{1-\rho} - 1\right] + \mu_n - c - \frac{1}{2}|\sigma_n|^2 + \mu_x \cdot \partial_x \upsilon + \frac{1-\gamma}{2}|\sigma_n|^2 \\ +(1-\gamma)\sigma_n \cdot \sigma_x' \partial_x \upsilon + \frac{1}{2}\operatorname{tr}\left(\sigma_x' \partial_{xx'} \upsilon \sigma_x\right) + \frac{1-\gamma}{2}|\sigma_x' \partial_x \upsilon|^2 \end{aligned} \tag{1}$$

Maximizing over consumption c delivers the following consumption-wealth ratios:

$$c^e(x) = \delta_e^{1/\rho_e} \exp\left(\left(1 - \frac{1}{\rho_e}\right)\upsilon^e(x)\right)$$
$$c^h(x) = \delta_h^{1/\rho_h} \exp\left(\left(1 - \frac{1}{\rho_h}\right)\upsilon^h(x)\right)$$

Next, note that the expert and household expected return-on-capital are given by

$$\mu_R^e = \frac{\alpha_e - i^*}{Q_t} + \mu_q + \Phi(i^*) + \beta_k z_1 - \eta_k + \sqrt{z_2}\sigma_k \cdot \sigma_q \tag{2}$$

$$\mu_R^h = \frac{\alpha_h - i^*}{Q_t} + \mu_q + \Phi(i^*) + \beta_k z_1 - \eta_k + \sqrt{z_2}\sigma_k \cdot \sigma_q \tag{3}$$

The only difference is the type-specific productivity α_j. The investment-to-capital ratio i^* only shows up in agents' HJBs through μ_R^j and it does so symmetrically. Thus, maximizing over i^* delivers

$$i^*(x) = (\Phi')^{-1}\left(\frac{1}{q(x)}\right) = \frac{q(x)-1}{\phi},$$

where the explicit form of i^* uses our specification of the installation function $\Phi(x) = \phi^{-1}\log(1+\phi i)$.

Finally, notice that the capital holding K^j and derivatives positions θ^j only show up in the HJB equation through the net worth drift and diffusion terms (μ_n, σ_n), leading to problem (26). Households maximize the portfolio problem (26) over all possible choices of their risk exposure vector $\sigma_{n,t}^h = \frac{Q_t K_t^h}{N_t^h}\sigma_{R,t} + \theta_t^h$ by choice of $K_t^h \geqslant 0$ and $\theta_t^h \in \mathbb{R}^3$, while experts maximize (26)) over all possible choices of $\sigma_{n,t}^e = \frac{Q_t K_t^e}{N_t^e}\sigma_{R,t} + \theta_t^e = \chi_t \frac{Q_t K_t^e}{N_t^e}\sigma_{R,t}$ such that $K_t^e \geqslant 0$ and $\chi_t \geqslant \underline{\chi}$. In the next subsections, we will solve these portfolio choice problems.

In this appendix, we will sometimes work with the risk premium "wedges" Δ^e and Δ^h, which are defined as the agent-specific gap between capital returns and market returns:

$$\Delta^e \stackrel{\text{def}}{=} \underline{\chi}^{-1}\big(\mu_R^e - r - \pi \cdot \sigma_R\big) \tag{4}$$

$$\Delta^h \stackrel{\text{def}}{=} \mu_R^h - r - \pi \cdot \sigma_R \tag{5}$$

We will also write households' and experts' shadow risk prices by π^h and π^e, respectively. Because households face complete markets, $\pi^h = \pi$ (i.e., their shadow risk price equals the traded risk price). Because experts face incomplete markets, $\pi^e \neq \pi$ generally speaking.

B.2 Household portfolio choice

The necessary conditions for optimality for households can be summarized as follows:

$$\begin{aligned} \mu_R^h - r + (1-\gamma_h)(\sigma_x \sigma_R) \cdot \partial_x \upsilon^h &\leqslant \gamma_h \sigma_R \cdot \sigma_n^h \\ \pi^h + (1-\gamma_h)\sigma_x' \partial_x \upsilon^h &= \gamma_h \sigma_n^h. \end{aligned}$$

Combining these equations, we have households' Euler equation,

$$\begin{cases} \mu_R^h - r \leqslant \pi^h \cdot \sigma_R, & \text{if} \quad K^h = 0 \\ \mu_R^h - r = \pi^h \cdot \sigma_R, & \text{if} \quad K^h > 0. \end{cases} \tag{6}$$

In other words, when households' expected capital return is below what they can earn with exposure to aggregate risk via futures contracts, they do not hold any capital. When they do hold capital, the expected return on such capital is equal to compensation for aggregate risk (via $\pi^h \cdot \sigma_R$). Households' optimal risk allocations are given by

$$\sigma_n^h = \frac{QK^h}{N^h}\sigma_R + \theta_h = \frac{\pi^h}{\gamma_h} + \frac{1-\gamma_h}{\gamma_h}\sigma_x' \partial_x \upsilon^h \tag{7}$$

B.3 Expert portfolio choice

Experts' portfolio choice is similar. Their first-order conditions

$$\begin{aligned} \mu_R^e - r - (1-\chi)\pi^h \cdot \sigma_R + \chi(1-\gamma_e)(\sigma_x\sigma_R) \cdot \partial_x \upsilon^e &= \gamma_e \chi \sigma_R \cdot \sigma_n^e \\ \pi^h \cdot \sigma_R + (1-\gamma_e)(\sigma_x\sigma_R) \cdot \partial_x \upsilon^e &\leqslant \gamma_e \sigma_R \cdot \sigma_n^e, \end{aligned}$$

can be combined to yield an Euler equation:

$$\begin{cases} \pi^h \cdot \sigma_R \leqslant \mu_R^e - r, & \text{if} \quad \chi = \underline{\chi} \\ \pi^h \cdot \sigma_R = \mu_R^e - r, & \text{if} \quad \chi > \underline{\chi}. \end{cases} \tag{8}$$

In other words, if the risk-premium $\pi_t^h \cdot \sigma_{R,t}$ required to be paid to the market for issuing equity is lower than the expected excess return that experts earn on their capital, they will issue as much equity as they can, and bounce against their skin-in-the-game constraint $\underline{\chi}$. Experts' optimal leverage is given by

$$\frac{\chi Q K^e}{N^e} = \frac{1}{\gamma_e |\sigma_R|^2}\Big[\Delta^e + \pi^h \cdot \sigma_R + (1-\gamma_e)(\sigma_x\sigma_R) \cdot \partial_x \upsilon^e\Big], \tag{9}$$

and because they take all aggregate risks in equal proportions, $\sigma_n^e = \frac{\chi Q K^e}{N^e}\sigma_R$. The "wedge" Δ_t^e is the incremental risk premium attained by experts, per unit of equity investment. To see that Δ^e represents an incremental private risk premium for experts, use the definition of Δ^e and experts' Euler equation to obtain the following equation: $\mu_R^e - r = \chi(\pi^h \cdot \sigma_R + \Delta^e) + (1-\chi)\pi^h \cdot \sigma_R$. In particular, $\chi(\pi^h \cdot \sigma_R + \Delta^e)$ represents the experts' excess return to "inside equity" whereas $(1-\chi)\pi^h \cdot \sigma_R$ represents the excess return to "outside equity" held by households. These sum to the excess return on assets, and Δ^e can thus be interpreted as the bonus return per unit of inside equity, of which there are χ units.

B.4 Equilibrium capital and risk distribution

Define expert's capital share

$$\kappa \stackrel{\text{def}}{=} \frac{K^e}{K}, \tag{10}$$

which fully summarizes the capital distribution. The Euler equations (6) and (8) can be used to determine κ and equity-retention share χ. Since households are less productive, it is often efficient for experts to manage all capital ($\kappa = 1$) and exhaust their equity-issuance capacity ($\chi = \underline{\chi}$). Thus, the solutions for κ and χ describe the nature of *occasionally-binding constraints* in this model.

Lemma B.1. *The equilibrium expert capital share κ_t and equity retention χ_t satisfy the following complementary slackness conditions:*

$$0 = \min(1 - \kappa_t, -\Delta_t^h) \tag{11}$$

$$0 = \min(\chi_t - \underline{\chi}, \Delta_t^e). \tag{12}$$

When households hold capital ($\kappa_t < 1$), experts are equity-issuance constrained ($\chi_t = \underline{\chi}$).

Lemma B.1 also shows how Δ^e and Δ^h play the roles of a Lagrange multipliers on equity-issuance and shorting constraints: Δ^e measures the shadow value of loosening the equity-issuance constraint (decreasing $\underline{\chi}$), whereas $-\Delta^h$ measures the shadow value of allowing households to short some of the capital stock.

Proof of Lemma B.1. Begin with Euler equations (6) and (8), use the definitions of Δ^h and Δ^e in (5)-(4), and use the definition of κ to immediately obtain (11)-(12). To verify the claim that $\kappa < 1$ implies $\chi = \underline{\chi}$, use the definitions of μ_R^e and μ_R^h which differ only in their dividend yields $(\alpha_e - i^*)/Q$ and $(\alpha_h - i^*)/Q$. Therefore, $\underline{\chi}\Delta^e = \Delta^h + \mu_R^e - \mu_R^h = \Delta^h + (\alpha_e - \alpha_h)/Q$. If $\kappa < 1$, then $\Delta^h = 0$ by (11). If $\Delta^h = 0$, then $\Delta^e > 0$ by the previous result, which implies $\chi = \underline{\chi}$ by (12). □

B.5 Price of capital

The goods market clearing condition can be written

$$\alpha_e K_t^e + \alpha_h K_t^h = C_t^e + C_t^h + I_t^e + I_t^h.$$

In other words, aggregate consumption plus aggregate investments by households and experts must equal aggregate output. Dividing the equation above by aggregate wealth $Q_t K_t$ in the economy, and remembering the definition of $\kappa = K^e/K$, we obtain

$$(1 - w)c^h + wc^e + \frac{i^*(q)}{q} = \frac{(1 - \kappa)\alpha_h + \kappa\alpha_e}{q} \tag{13}$$

Equation (13) relates q and κ to the state variables, conditional on knowing the wealth-normalized value functions υ^h and υ^e. One can show that this equation in q has a unique positive root. We also notice that in the unitary IES case, when all the capital in the economy is held by experts (i.e. $\kappa = 1$), the price of capital is invariant to the driving processes (z_1, z_2), and simply equal to the ratio of the dividend yield $\alpha_e - i^*(q)$ divided by the wealth-weighted average rate of time preference $w\delta_e + (1-w)\delta_h$. With our functional form assumed for the installation function Φ, we obtain the following price of capital:

$$q = \frac{(1-\kappa)\alpha_h + \kappa\alpha_e + 1/\phi}{(1-w)\delta_h^{1/\rho_h}\exp\left((1-\frac{1}{\rho_h})v^h\right) + w\delta_e^{1/\rho_e}\exp\left((1-\frac{1}{\rho_e})v^e\right) + 1/\phi} \tag{14}$$

B.6 Law of motion of K, W, and Q

Because experts and households utilize a common investment rate $i^*(q)$, aggregate capital dynamics are particularly simple:

$$\frac{dK_t}{K_t} = \mu_{K,t}dt + \sigma_{K,t}\cdot dB_t \tag{15}$$

$$\mu_{K,t} \doteq \beta_k Z_t^1 + \Phi[i^*(Q_t)] - \eta_k \tag{16}$$

$$\sigma_{K,t} \doteq \sqrt{Z_t^2}\sigma_k \tag{17}$$

The law of motion of the wealth distribution W is derived below. Note in deriving these equations, we are allowing for an overlapping generations (OLG) structure with a birth/death rate of λ_d and a fraction ν of newborns exogenously designated experts. Dying agent wealth is automatically redistributed to newborns on a per-capita basis.

Lemma B.2. *The drift $\mu_{w,t}$ and diffusion $\sigma_{w,t}$ of the wealth share W_t are given by*

$$\mu_{w,t} = W_t(1-W_t)\Big[c_t^h - c_t^e + \frac{\chi_t\kappa_t}{W_t}\Delta_t^e\Big] + \sigma_{w,t}\cdot(\pi_t^h - \sigma_{R,t}) + \lambda_d(\nu - W_t) \tag{18}$$

$$\sigma_{w,t} = (\chi_t\kappa_t - W_t)\,\sigma_{R,t}. \tag{19}$$

Proof of Lemma B.2. Combine agents' dynamic budget constraints with their portfolio choices to obtain the evolution of aggregate households' and aggregate experts' wealth N_t^h and N_t^e:

$$\frac{dN_t^h}{N_t^h} = \Big[r_t - c^h - \lambda_d + \sigma_{n,t}^h\cdot\pi_t^h + \frac{1-\kappa_t}{1-W_t}\Delta_t^h + \frac{(1-\nu)\lambda_d}{1-W_t}\Big]dt + \sigma_{n,t}^h\cdot dB_t \tag{20}$$

$$\frac{dN_t^e}{N_t^e} = \Big[r_t - c^e - \lambda_d + \sigma_{n,t}^e\cdot\pi_t^h + \frac{\chi_t\kappa_t}{W_t}\Delta_t^e + \frac{\nu\lambda_d}{W_t}\Big]dt + \sigma_{n,t}^e\cdot dB_t, \tag{21}$$

and where

$$\sigma_n^h = \frac{1-\chi\kappa}{1-w}\sigma_R \tag{22}$$

$$\sigma_n^e = \frac{\chi\kappa}{w}\sigma_R. \tag{23}$$

The terms containing λ_d represent contributions from OLG. The key observation in obtaining the risk exposures (e.g., terms involving $\chi\kappa$ and $1-\chi\kappa$) is that experts hold $\chi\kappa$ fraction of total capital risk in the economy (after equity-issuance), so households must hold the balance $1-\chi\kappa$ by market clearing. The terms involving Δ^h and Δ^e come from recalling their definitions along with that of κ and w; e.g., households earn excess return Δ^h on their capital holdings, which equal $\frac{1-\kappa}{1-w}$ per unit of their net worth.

By Itô's formula, the wealth share $W_t = \frac{N_t^e}{N_t^e+N_t^h}$ evolves as

$$dW_t = W_t(1-W_t)\left(\frac{dN_t^e}{N_t^e}-\frac{dN_t^h}{N_t^h}\right)-W_t(1-W_t)\left(W_t\frac{d[N_t^e]}{(N_t^e)^2}-(1-W_t)\frac{d[N_t^h]}{(N_t^h)^2}+(1-2W_t)\frac{d[N_t^e,N_t^h]}{N_t^eN_t^h}\right)$$

Using (20)-(21) and (22)-(23), and making several simplifications, the result is

$$\mu_w = w(1-w)\left[c^h - c^e + \frac{\chi\kappa}{w}\Delta^e - \frac{1-\kappa}{1-w}\Delta^h\right] + (\chi\kappa - w)\sigma_R\cdot(\pi^h - \sigma_R) + \lambda_d(\nu - w) \tag{24}$$

$$\sigma_w = (\chi\kappa - w)\sigma_R. \tag{25}$$

The result of Lemma B.2 is obtained by using Lemma B.1 to get $(1-\kappa)\Delta^h = 0$. □

Finally, by Itô's formula, the drift and diffusion coefficients of Q_t are

$$\mu_q = \mu_x\cdot\partial_x\log q + \frac{1}{2}\left[\mathrm{tr}\left(\sigma_x'\partial_{xx'}\log(q)\sigma_x\right) + |\sigma_x'\partial_x\log q|^2\right] \tag{26}$$

$$\sigma_q = \sigma_x'\partial_x\log q. \tag{27}$$

On the other hand, σ_x depends on σ_q, constituting a two-way feedback loop. We can solve this loop by substituting the expression for σ_x into the formula for σ_q, using $\sigma_R = \sqrt{z_2}\sigma_k + \sigma_q$ to obtain:

$$\sigma_q = \frac{(\chi\kappa - w)\left(\partial_w\log q\right)\sqrt{z_2}\sigma_k + \sigma_z'\partial_z\log q}{1-(\chi\kappa-w)\partial_w\log q}, \tag{28}$$

where recall $z \stackrel{\text{def}}{=} (z_1, z_2)'$ and $\sigma_z = \sqrt{z_2}(\sigma_1, \sigma_2)'$. Conditional on knowing χ and κ, if we know the price function q across the state space, we know the capital price volatility vector σ_q, as well as the wealth share volatility vector σ_w. Note that this generates capital return volatility equal to

$$\sigma_R = \frac{\sqrt{z_2}\sigma_k + \sigma_z' \partial_z \log q}{1 - (\chi\kappa - w)\partial_w \log q}. \tag{29}$$

B.7 Equilibrium risk-free rate and risk prices

We solve for the risk-free rate r as well as the households' and experts' risk-prices π^h, π^e. To do this, we use the fact that $Q_t K_t = N_t^h + N_t^e$, which we time-differentiate. Using the dynamic evolution equations for N^h and N^e in (20) and (21), and for K in (15), by equating the drift terms we obtain:

$$r + (1-w)\Big(\sigma_n^h \cdot \pi^h + \frac{1-\kappa}{1-w}\Delta^h - c^h\Big) + w\Big(\sigma_n^e \cdot \pi^h + \frac{\chi\kappa}{w}\Delta^e - c^e\Big) = \mu_q + \mu_K + \sigma_K \cdot \sigma_q \tag{30}$$

By equating the diffusion terms:

$$(1-w)\sigma_n^h + w\sigma_n^e = \sigma_R. \tag{31}$$

To solve for r, substitute (31) into (30), use the result from Lemma B.1 that $(1-\kappa)\Delta^h = 0$, and rearrange:

$$r = \mu_q + \mu_K + \sigma_K \cdot \sigma_q - \sigma_R \cdot \pi^h + wc^e + (1-w)c^h - \chi\kappa\Delta^e \tag{32}$$

To solve for π^h, substitute optimal exposure σ_n^h from (7) with its equilibrium value from (22) to obtain:

$$\pi^h = \gamma_h \frac{1-\chi\kappa}{1-w}\sigma_R + (\gamma_h - 1)\sigma_x' \partial_x \upsilon^h. \tag{33}$$

Since experts face incomplete markets, there is in theory an infinite number of stochastic discount factors that can price claims for which the expert is marginal. We thus focus on the marginal utility of consumption process, which for any agent with recursive preferences takes the following form (see for example Duffie and Epstein (1992)):

$$S_t = \exp\left[\int_0^t \left(\left(\frac{\rho-\gamma}{1-\rho}\right)\delta^{1/\rho}\exp\left(\left(1-\frac{1}{\rho}\right)\upsilon_s\right) - \delta\left(\frac{1-\gamma}{1-\rho}\right)\right) ds\right] N_t^{-\gamma}\exp\left((1-\gamma)\upsilon_t\right)$$

In the case of time- and state-separability (i.e. when $\rho=\gamma$), we obtain the familiar formula $S_t/S_0 = e^{-\delta t}\left(C_t/C_0\right)^{-\gamma}$. Remember that we have for households and experts:

$$\frac{dN_t^j}{N_t^j} = \left(\mu_{n,t}^j - c_t^j\right) dt + \sigma_{n,t}^j \cdot dB_t$$

This leads to the key equation defining the vector of shadow risk prices faced by an investor:

$$\frac{dS_t^j}{S_t^j} - \mathbb{E}_t\left[\frac{dS_t^j}{S_t^j}\right] = -\left[\gamma_j\sigma_{n,t}^j + (\gamma_j - 1)\sigma_{\upsilon,t}^j\right]\cdot dB_t \stackrel{\text{def}}{=} -\pi_t^j \cdot dB_t \tag{34}$$

In the above, the k^{th} coordinate of π_t^j is the expected excess return investor j gets paid per unit of risk exposure to the k^{th} shock of B_t. Substituting formula (23) for σ_n^e into (34), we have

$$\pi^e = \gamma_e\frac{\chi\kappa}{w}\sigma_R + (\gamma_e - 1)\sigma_x'\partial_x\upsilon^e \tag{35}$$

This means that experts' equilibrium expected excess return compensation is equal to:

$$\begin{aligned}\pi^e\cdot\sigma_R &= \gamma_e\frac{\chi\kappa}{w}|\sigma_R|^2 + (\gamma_e - 1)\left(\sigma_x\sigma_R\right)\cdot\partial_x\upsilon^e \\ &= \Delta^e + \pi^h\cdot\sigma_R,\end{aligned}$$

where π^h is the vector of aggregate risk prices faced by households.[1]

[1]Note that we could follow these same steps for households but would obtain an equivalent result to our equilibrium risk price vector. Substitute formula (22) for σ_n^h into the shadow risk-price definition (34) to get

$$\pi^h = \gamma_h\frac{1-\chi\kappa}{1-w}\sigma_R + (\gamma_h - 1)\sigma_x'\partial_x\upsilon^h \tag{36}$$

Notice that π^h in (36) is identical to (33).

B.8 Deriving the functional equation for χ

We first note that the experts' aggregate risk choice (9) can be re-arranged to express the expected return premium Δ^e as follows:

$$\Delta^e = \gamma_e \frac{\chi\kappa}{w}|\sigma_R|^2 - \gamma_h \frac{1-\chi\kappa}{1-w}|\sigma_R|^2 - (\sigma_x \sigma_R) \cdot \big[(\gamma_h - 1)\partial_x \upsilon^h - (\gamma_e - 1)\partial_x \upsilon^e\big] \tag{37}$$

Substituting (37) into the complementary slackness condition for experts' skin-in-the-game constraint (12),

$$0 = \min\Big\{\chi - \underline{\chi}, (1-w)\gamma_e \chi\kappa|\sigma_R|^2 - w\gamma_h(1-\chi\kappa)|\sigma_R|^2 \\ - w(1-w)\,(\sigma_x\sigma_R)\cdot\big[(\gamma_h-1)\partial_x\upsilon^h - (\gamma_e-1)\partial_x\upsilon^e\big]\Big\}.$$

Since all the capital is held by experts whenever their skin-in-the-game constraint is not binding, we may substitute $\kappa = 1$ everywhere in this equation, as shown in Lemma B.1. The above equation is actually an algebraic equation for χ, which can be solved by substituting σ_x and σ_R into the second term in the minimum, obtaining

$$0 = \min\Big\{\chi - \underline{\chi}, \Big[((1-w)\gamma_e + w\gamma_h)|D_z|^2 + (\partial_w \log q)D_{\upsilon,z} - D_{\upsilon,w}\Big](\chi - w) \\ + w(1-w)(\gamma_e - \gamma_h)|D_z|^2 - D_{\upsilon,z}\Big\}. \tag{38}$$

In the above, we have defined[2]

$$D_z \doteq \sqrt{z_2}\sigma_k + \sigma_z'\partial_z \log q \tag{39}$$

$$D_{\upsilon,w} \doteq w(1-w)|D_z|^2\partial_w\big[(\gamma_h-1)\upsilon^h - (\gamma_e-1)\upsilon^e\big] \tag{40}$$

$$D_{\upsilon,z} \doteq w(1-w)\,(\sigma_z D_z)\cdot\partial_z\big[(\gamma_h-1)\upsilon^h - (\gamma_e-1)\upsilon^e\big]. \tag{41}$$

When $\chi > \underline{\chi}$, the second term of the minimum operator in equation (38) is linear in $\chi - w$, holding fixed the functions $(q, \upsilon^e, \upsilon^h)$.

The analysis is simpler in one special case. If risk aversions are identical ($\gamma_e = \gamma_h = \gamma$),

[2]The notation above is helpful, since it allows us to write $|D_z|^2 = (1 - (\chi\kappa - w)\,\partial_w \log q)^2\,|\sigma_R|^2$, and simplify the expression for $\sigma_x\sigma_R$ as follows:

$$\sigma_x\sigma_R = \frac{1}{1-(\chi\kappa - w)\,\partial_w\log q}\begin{pmatrix}\frac{\chi\kappa - w}{1-(\chi\kappa-w)\partial_w\log q}|D_z|^2 \\ \sigma_z D_z\end{pmatrix}$$

the second term of the minimum operator in equation (38) simplifies substantially. Then, we may prove the following proposition, which says for many parameters that the skin-in-the-game constraint is either always-binding or never-binding.

Proposition B.3. *Suppose agents have identical risk aversions ($\gamma_e = \gamma_h = \gamma$). Experts' optimal risk retention χ is*

$$\chi = \max(\underline{\chi}, w). \tag{42}$$

For $w \geqslant \underline{\chi}$, the skin-in-the-game constraint is slack and the wealth share evolves (locally) deterministically, i.e., $\sigma_w = 0$. At $w = \underline{\chi}$, when the skin-in-the-game constraint just binds, the formula for the drift of the expert wealth share is

$$\mu_w(\underline{\chi}, z) = \underline{\chi}(1 - \underline{\chi})\left[c^h(\underline{\chi}, z) - c^e(\underline{\chi}, z)\right] + \lambda_d(\nu - \underline{\chi}).$$

The following hold:

(i) If $W_t \leqslant \underline{\chi}$ for some time t and $\sup_z \mu_w(\underline{\chi}, z) < 0$, then $\chi_t = \underline{\chi}$ with probability one.

(ii) If $W_t \geqslant \underline{\chi}$ for some time t and $\inf_z \mu_w(\underline{\chi}, z) > 0$, then $\chi_t > \underline{\chi}$ with probability one.

Proof of Proposition B.3. If $\gamma_e = \gamma_h = \gamma$, then the second term in the minimum operator in equation (38) becomes

$$\mathcal{M}(\chi) \doteq \Big[\gamma|D_z|^2 + (\partial_w \log q)D_{v,z} - D_{v,w}\Big](\chi - w) - D_{v,z}. \tag{43}$$

Whenever $\chi > \underline{\chi}$, we solve for χ from $\mathcal{M}(\chi) = 0$. Therefore, $\chi = \max(\underline{\chi}, \chi^*)$, where $\chi^* \in \{y : \mathcal{M}(y) = 0\}$. As a preliminary, we show that $\chi^* = w$ solves $\mathcal{M}(\chi^*) = 0$, such that (42) holds. To prove this, conjecture (and later verify) that $\pi^e = \pi^h$ on $\chi > \underline{\chi}$. Using (35) and (36), this conjecture implies

$$\gamma\frac{\chi\kappa}{w}\sigma_R + (\gamma - 1)\sigma_x'\partial_x v^e = \gamma\frac{1 - \chi\kappa}{1 - w}\sigma_R + (\gamma - 1)\sigma_x'\partial_x v^h, \quad \text{if} \quad \chi > \underline{\chi}. \tag{44}$$

Since $\kappa = 1$ when $\chi > \underline{\chi}$ (Lemma B.1), and since $\sigma_w = (\chi\kappa - w)\sigma_R = (\chi - w)\sigma_R$, equation (44) reduces to

$$(\gamma - 1)\sigma_z'\partial_z\left(v^h - v^e\right) = (\chi - w)\frac{\gamma - (\gamma - 1)w(1 - w)\partial_w(v^h - v^e)}{w(1 - w)}\sigma_R, \quad \text{if} \quad \chi > \underline{\chi}. \tag{45}$$

Substituting (45) into (41) and (43), we obtain

$$\mathcal{M}(\chi) = (\chi - w)\Big[\gamma|D_z|^2 + (\partial_w \log q)D_{\upsilon,z} - D_{\upsilon,w} - \Big(\gamma - (\gamma-1)w(1-w)\partial_w(\upsilon^h - \upsilon^e)\Big)D_z'\sigma_R\Big].$$

Consequently, $\chi^* = w$ is one solution to $\mathcal{M}(\chi^*) = 0$.

Under this solution, we may verify $\pi^e = \pi^h$ as follows. First, use $\sigma_w = (\chi - w)\sigma_R = 0$ when $\chi = \chi^* = w$ to find from (22)-(23) that this equilibrium features

$$\sigma_n^e = \sigma_n^h = \sigma_R, \quad \text{if} \quad \chi > \underline{\chi}. \tag{46}$$

Second, introduce to all agents, for a short period of time, zero-net-supply Arrow-Debreu claims on each of the Brownian shocks. Let σ_n^{e*} and σ_n^{h*} denote agents' risk exposures in this modified economy. In the modified equilibrium, there is a single traded risk price π^* on these shocks, and both expert and household risk prices coincide with π^*. Also, since these Arrow-Debreu assets are only introduced for an arbitrarily short period of time, agents value processes υ^e and υ^h are unaffected. Putting these results together, and using formulas (35) and (36), we have

$$\begin{aligned}
\gamma\sigma_n^e &= \pi^e + (1-\gamma)\sigma_x'\partial_x\upsilon^e \\
\gamma\sigma_n^h &= \pi^h + (1-\gamma)\sigma_x'\partial_x\upsilon^h \\
\gamma\sigma_n^{e*} &= \pi^* + (1-\gamma)\sigma_x'\partial_x\upsilon^e \\
\gamma\sigma_n^{h*} &= \pi^* + (1-\gamma)\sigma_x'\partial_x\upsilon^h.
\end{aligned}$$

By repeating the arguments leading to (46), we know that the modified equilibrium also features $\sigma_n^{e*} = \sigma_n^{h*} = \sigma_R$. Therefore, $\pi^e = \pi^h = \pi^*$.

Finally, because $\mathcal{M}(\chi^*) = 0$ in (43) is a linear equation in χ^*, it admits a unique solution, so $\chi^* = w$ must be the only solution. This proves (42).

Next, to demonstrate cases (i) and (ii), compute the drift μ_w. We have already shown that $\sigma_w = 0$ when $\chi > \underline{\chi}$, so it suffices to show that $\mu_w < 0$ when $\chi \geqslant \underline{\chi}$ in case (i) and $\mu_w > 0$ when $\chi \leqslant \underline{\chi}$ in case (ii). For case (i), the condition $W_t \leqslant \underline{\chi}$ implies that we need only show $\mu_w < 0$ when $\chi = \underline{\chi}$. Similarly for case (ii), the condition $W_t \geqslant \underline{\chi}$ implies that we need only show $\mu_w > 0$ when $\chi = \underline{\chi}$. These are implied by $\sup_z \mu_w(\underline{\chi}, z) < 0$ and $\inf_z \mu_w(\underline{\chi}, z)$, respectively. □

B.9 Deriving the functional equation for κ

First, note that (37) is an equation relating χ, κ, and Δ^e. Second, by taking the difference $\mu_R^e - \mu_R^h = \mu_R^e - r - \pi \cdot \sigma_R + \pi \cdot \sigma_R - (\mu_R^h - r)$, using the definitions of μ_R^e and μ_R^h, along with the definitions of Δ^e and Δ^h in (4)-(5), we obtain:

$$\Delta^h = \underline{\chi}\Delta^e - \frac{a_e - a_h}{q}. \tag{47}$$

Now, combine the complementary-slackness condition for households' capital holdings (11) from Lemma B.1, with (37) and (47) to obtain

$$\begin{aligned} 0 = \min\Big\{ & 1-\kappa,\, w\gamma_h(1-\chi\kappa)|\sigma_R|^2 - (1-w)\gamma_e\chi\kappa|\sigma_R|^2 \\ & + w(1-w)\frac{\alpha_e - \alpha_h}{\underline{\chi} q} + w(1-w)\left(\sigma_x\sigma_R\right)\cdot\left[(\gamma_h - 1)\partial_x \upsilon^h - (\gamma_e - 1)\partial_x \upsilon^e\right]\Big\}. \end{aligned} \tag{48}$$

We may substitute $\chi = \underline{\chi}$ everywhere in this equation, due to Lemma B.1. Given (υ^e, υ^h), equation (48) is actually a standalone variational inequality (differential equation wrapped inside of a min operator) for κ, since q can be expressed solely as a function of $(\kappa, \upsilon^e, \upsilon^h)$ through (13), and since both σ_x and σ_R can be expressed solely in terms of χ, κ, q, and $\partial_x q$ through (25) and (29). By inspection, the boundary condition $\kappa(0, z) = 0$ will be satisfied automatically as long as $\alpha_h > -\infty$.

B.10 Asymptotic analysis as $w \to 0$

In this section, we work in a one-dimensional model (no shocks to Z), so assume $\sigma_z = 0$. We will allow the birth/death (OLG) process with rate λ_d but will assume all newborn agents are born as households, so $\nu = 0$. We will make following assumption on the nature of equilibrium and analyze the two cases separately.

Assumption B.4. *One of the following two assumptions hold as $w \to 0$. Either (i) $\chi\kappa/w \to C \in (1, \infty)$ or (ii) $\chi\kappa \to C \in (0, 1]$.*

Our goal is to prove the following proposition:

Proposition B.5. *Suppose $\rho_e = \rho_h = 1$, $\nu = 0$, $\delta_e \geqslant \delta_h$, and $\gamma_h \geqslant \gamma_e$. Shut down growth and volatility shocks, $\sigma_z = 0$. Models satisfying case (ii) of Assumption B.4 feature a* *stationary wealth density that decays quadratically, i.e.,*

$$f(w) \sim G_0 w^2 \quad as \quad w \to 0, \quad some\ constant \quad G_0.$$

Models satisfying case (i) of Assumption B.4 feature a stationary wealth density that decays at rate ζ, i.e.,

$$f(w) \sim G_0 w^\zeta \quad as \quad w \to 0, \quad some\ constant \quad G_0,$$

where

$$\zeta \stackrel{\text{def}}{=} \frac{2[\delta_h - \delta_e - \lambda_d + (\gamma_e C^2 - \gamma_h - (C-1))\sigma_k^2]}{(C-1)^2\sigma_k^2} - 2,$$

and where $C \geqslant 1$ is given below in equations (51) or (52), depending on parameters. Consequently, the lower tail of models in case (i) is thicker than that of case (ii) models if and only if $\delta_h - \delta_e - \lambda_d < [(C-1)(2C-1) - \gamma_e C^2 + \gamma_h]\sigma_k^2$.

Remark 1. *Proposition B.5 imposes $\nu = 0$ (experts are never exogenously "reborn") in order to make a stark comparison between two classes of models. If $\nu > 0$, then the formula for the tail index ζ will change, because these economies feature $\mu_w(0) = \nu\lambda_d > 0$ and $\sigma_w(0) = 0$. A particular implication is that, if $\nu > 0$, the density $f(w)$ can never have an asymptote as $w \to 0$, whereas an asymptote is possible if $\nu = 0$. That said, the point of Proposition B.5 is to provide guidance on features that generically "thicken" the tail of the wealth share density, and these features remain the same in economies with $\nu > 0$.*

Proof of Proposition B.5. Below, we will use the notation $g_1(w) \sim g_2(w)$ to mean $g_1(w)/g_2(w) \to 1$ as $w \to 0$. For expedience, we assume, but do not verify (although it can be verified), that $\mu_q \sim \bar{\mu}_q$ and $\sigma_q \sim \bar{\sigma}_q$ for bounded constants $\bar{\mu}_q$ and $\bar{\sigma}_q$.[3] This assumption, in particular, implies $\sigma_R \sim \sigma$ for some constant σ. From Lemma B.2 and the form of π^e, π^h in (35) and (36), we have the following asymptotic state dynamics:

$$\mu_w \sim (\delta_h - \delta_e - \lambda_d)w + \chi\kappa\sigma^2[\gamma_e \frac{\chi\kappa}{w} + (\gamma_e - 1)(\chi\kappa - w)\frac{d}{dw}\upsilon^e - \gamma_h(1-\chi\kappa) - (\gamma_h - 1)(\chi\kappa - w)\frac{d}{dw}\upsilon^h]$$
$$+ (\chi\kappa - w)\sigma^2[\gamma_h(1-\chi\kappa) + (\gamma_h - 1)(\chi\kappa - w)\frac{d}{dw}\upsilon^h - 1] \quad (49)$$

[3]In either case, we would conjecture $q \sim A_q + B_q w$ and derive the aforementioned facts by solving A_q and B_q both from goods market clearing, e.g., (14). After determining A_q and B_q, the values of $\bar{\mu}_q$ and $\bar{\sigma}_q$ could be obtained by applying Itô's formula to q.

$$\sigma_w \sim (\chi\kappa - w)\sigma \tag{50}$$

In the one-dimensional model, agents' HJB equations take the following form:

$$0 = \max\left\{\delta(\log\delta - 1 - \upsilon) + \mu_n - \frac{\gamma}{2}\sigma_n^2 + [\mu_w + (1-\gamma)\sigma_n\sigma_w]\upsilon' + \frac{1}{2}\sigma_w^2\upsilon'' + \frac{1-\gamma}{2}\sigma_w^2(\upsilon')^2\right\},$$

where

$$\begin{aligned}
\mu_n^e &\sim r + \frac{\chi\kappa}{w}\Big[\gamma_e(\chi\kappa/w) + (\gamma_e - 1)(\chi\kappa - w)\frac{d}{dw}\upsilon^e\Big]\sigma^2 \\
\mu_n^h &\sim r + (1-\chi\kappa)\Big[\gamma_h(1-\chi\kappa) + (\gamma_h - 1)(\chi\kappa - w)\frac{d}{dw}\upsilon^h\Big]\sigma^2 \\
\sigma_n^e &\sim \frac{\chi\kappa}{w}\sigma \quad \text{and} \quad \sigma_n^h \sim (1-\chi\kappa)\sigma
\end{aligned}$$

Now, we consider the two cases of Assumption B.4.

Case 1: $\chi\kappa/w \to C$. Conjecture state dynamics of the asymptotic form

$$\mu_w \sim B_\mu w \quad \text{and} \quad \sigma_w \sim B_\sigma w.$$

Conjecture also value functions take the asymptotic form

$$\upsilon^e \sim A_e + B_e w^{\zeta_e} \quad \text{and} \quad \upsilon^h \sim A_h + B_h w^{\zeta_h} \quad \text{where} \quad \zeta_e, \zeta_h > 0.$$

Substituting these assumptions into (49)-(50) shows that

$$\begin{aligned}
B_\mu &= \delta_h - \delta_e - \lambda_d + [\gamma_e C^2 - \gamma_h - (C-1)]\sigma^2 \\
B_\sigma &= (C-1)\sigma
\end{aligned}$$

At this point, we have the dynamics of w, independently of the value functions, but we still must verify the conjecture.

Asymptotically, the HJBs require the following dominant-term equations to hold,

$$\begin{aligned}
0 &= \delta_e[\log\delta_e - 1 - A_e] + r + \gamma_e C^2\sigma^2 - \frac{\gamma_e}{2}C^2\sigma^2 \\
0 &= \delta_h[\log\delta_h - 1 - A_h] + r + \gamma_h\sigma^2 - \frac{\gamma_h}{2}\sigma^2.
\end{aligned}$$

We substitute r which must take the form

$$r \sim \delta_h + \Phi(i^*(q)) - \eta_k + \bar{\mu}_q + \bar{\sigma}_q\sigma_k - \gamma_h\sigma^2,$$

which is bounded (and equals the representative-agent risk-free rate when households dominate the economy). Substituting this into the HJB equations, we obtain explicit expressions for A_e, A_h. For the terms of order w^{ζ_e}, w^{ζ_h}, the HJB equations say

$$\delta_e = B_\mu\zeta_e + \frac{1}{2}B_\sigma^2\zeta_e(\zeta_e - 1)$$
$$\delta_h = B_\mu\zeta_h + \frac{1}{2}B_\sigma^2\zeta_h(\zeta_h - 1),$$

whereby B_e, B_h have dropped out (these constants are determined by the right boundary $w = 1$). These quadratic equations each have one positive and one negative root. Taking the positive root, we verify that $\zeta_e, \zeta_h > 0$.

It remains to determine C. The solution depends on the separate asymptotics of κ and χ, not only their product. If $\kappa \to 0$ while $\chi \to \underline{\chi} \neq 0$, then the following analysis holds. Using the definition of μ_R^j, the relationship $\mu_R^e = \sigma_R \cdot [\chi\pi^e + (1-\chi)\pi^h]$, and previous results on asymptotics as $w \to 0$, we have that

$$\frac{\alpha_e - \alpha_h}{q} \sim \chi(\gamma_e C - \gamma_h)\sigma^2.$$

Using the goods market clearing condition (13), we see that $q(0)$ is independent of C. Hence,

$$C = \frac{\gamma_h}{\gamma_e} + \frac{\alpha_e - \alpha_h}{\underline{\chi}\sigma^2 q(0)} \geqslant 1. \tag{51}$$

On the other hand, if $\kappa \to 1$ while $\chi \to 0$ (e.g., this occurs if $\alpha_h = -\infty$ and $\underline{\chi} = 0$), then we may use equation (38) to obtain[4]

$$C = \frac{\gamma_h}{\gamma_e} \geqslant 1. \tag{52}$$

The cases of interest, where $C > 1$ strictly, are when either (a) $\gamma_h > \gamma_e$ and $\alpha_e = \alpha_h$ as in Gârleanu and Panageas (2015); or (b) $\gamma_h = \gamma_e$ and $\alpha_e > \alpha_h$ as in Brunnermeier and

[4] Note that these equations for C agree if $\alpha_e = \alpha_h$ and $\underline{\chi} = 0$, which shows that a frictionless economy in the spirit of Gârleanu and Panageas (2015) can be implemented in our model by equivalently allowing one of either χ or κ to adjust.

Sannikov (2014).

Returning to the dynamics of w, we have the Kolmogorov Forward Equation, which reads

$$0 = -\frac{d}{dw}[\mu_w f] + \frac{1}{2}\frac{d^2}{dw^2}[\sigma_w^2 f].$$

Integrating from an interior point w to 1, and using the fact that $\mu_w(1) < 0$ and $\sigma_w(1) = 0$ in all models we consider, implying f and σ_w both vanish at the upper boundary, we obtain

$$0 = -\mu_w f + \frac{1}{2}\frac{d}{dw}[\sigma_w^2 f].$$

Asymptotically, as $w \to 0$, we have

$$0 = -B_\mu w f + \frac{1}{2}B_\sigma \frac{d}{dw}[w^2 f] + o(w).$$

Solving this equation shows that, asymptotically,

$$f(w) \sim G_0 w^{2(B_\mu/B_\sigma^2 - 1)}, \quad \text{some constant} \quad G_0 > 0. \tag{53}$$

This equation determines the existence (non-degeneracy) and asymptotic shape of the stationary density.[5]

Case 2: $\chi\kappa \to C$**.** In this case, it suffices to consider parameters $\underline{\chi} > 0$ and $\alpha_h = -\infty$, in which case $\chi \sim \underline{\chi}$ and $\kappa \sim 1$. Then, $C = \underline{\chi} \in (0, 1]$ as desired. Mimicking the previous analysis, conjecture that

$$\mu_w \sim B_\mu/w \quad \text{and} \quad \sigma_\mu \sim A_\sigma$$

and for the value functions

$$\upsilon^e \sim A_e + B_e \log(w) \quad \text{and} \quad \upsilon^h \sim A_h + B_h w.$$

[5] As long as $2B_\mu > B_\sigma^2$, a non-degenerate density exists. This condition is

$$2[\delta_h - \delta_e - \lambda_d] > [3(C-1)^2 - 2\gamma_e C^2 + 2\gamma_h]\sigma^2.$$

The shape is given by the exponent $2(B_\mu/B_\sigma^2 - 1)$. If, as in Brunnermeier and Sannikov (2014), $\frac{1}{2}B_\sigma^2 < B_\mu < B_\sigma^2$, then the density has an asymptotic spike.

These conjectures imply

$$\begin{aligned}A_\sigma &= C\sigma\\B_\mu &= (C\sigma)^2[\gamma_e + (\gamma_e - 1)B_e]\\\pi^e &\sim (C\sigma/w)[\gamma_e + (\gamma_e - 1)B_e]\\\pi^h &\sim \sigma[(1-C)\gamma_h + C(\gamma_h - 1)B_h]\\r &\sim A_r + (1/w)B_r\end{aligned}$$

for constants $A_r \doteq \delta_h + \Phi(i^*(q(0))) - \eta_k + \frac{q'(0)}{q(0)}A_\sigma\sigma_k - (1-C)\sigma[(1-C)\gamma_h + C(\gamma_h - 1)B_h]$ and

$$B_r \doteq \frac{q'}{q}B_\mu - (C\sigma)^2(\gamma_e + (\gamma_e - 1)B_e)].$$

Note that equation (13) with $\kappa = 1$ shows that q'/q is bounded. Indeed, we have

$$\frac{q'(0)}{q(0)} = -\frac{\delta_e - \delta_h}{\delta_h + 1/\phi_1}, \quad \text{some constant} \quad \phi_1 > 0.$$

which is the same equation one would obtain for the specific functional form leading to (14). In the above, ϕ_1 is to be interpreted as the local elasticity of the accumulation function Φ near $w \sim 0$, whereas this elasticity is assumed globally constant in (14).

Substituting these results into agents' HJB equations, and keeping only the highest-order terms, we obtain

$$\begin{aligned}0 &= \frac{1}{w^2}\Big[\gamma_e(C\sigma)^2 - \frac{1}{2}\gamma_e(C\sigma)^2 + B_\mu B_e - \frac{1}{2}A_\sigma^2 B_e + \frac{1-\gamma_e}{2}A_\sigma^2 B_e^2\Big]\\0 &= \frac{1}{w}\Big[B_r + B_\mu B_h\Big].\end{aligned}$$

Due to $1/w \to \infty$ as $w \to 0$, the terms in brackets must be 0 for the equations to hold. Substituting previous results and simplifying, we obtain[6]

$$\begin{aligned}B_e &= -1\\B_h &= 1 + \frac{\delta_e - \delta_h}{\delta_h + 1/\phi_1}.\end{aligned}$$

Any A_e, A_h are consistent with the HJBs at this boundary.

[6]Note that B_e solves a quadratic equation $0 = \gamma_e + (2\gamma_e - 1)B_e + (\gamma_e - 1)B_e^2$, which has the second solution $B_e = -\gamma_e/(\gamma_e - 1)$. However, substituting this root yields $\pi^e \sim 0$.

The state dynamics are thus given by

$$\mu_w \sim (C\sigma)^2(1/w) \quad \text{and} \quad \sigma_w \sim C\sigma.$$

Hence, repeating the same analysis of the Kolmogorov Forward Equation as in case 1, we obtain asymptotically,

$$f(w) \sim G_0 w^2, \quad \text{some constant} \quad G_0 > 0. \tag{54}$$

Thus, the density has a tail that decays quadratically, irrespective of B_μ, B_σ and by extension the model parameters.

Comparing the cases. Comparing the formulas (53) and (54), we see that case 1 has a thicker tail than case 2, if and only if

$$\delta_h - \delta_e - \lambda_d < [(C-1)(2C-1) - \gamma_e C^2 + \gamma_h]\sigma^2,$$

where C is given either by (51) or (52) depending on the context. This analysis proves Proposition B.5. □

C Computational Appendix

Joseph Huang, Haomin Qin and Chun Hei Hung[7]

C.1 Global solutions using finite difference methods

We solve the single-agent models in Sections 4 as well as the heterogeneous agents model in environment IP in Section 5 using finite difference methods. The single-agent models require solving a PDE of the following form:

$$L_{HJB}\left(v;x\right)=0 \tag{55}$$

Where $x=(z^1,z^2)$ in Section 4.4, $x=(z^1)$ in Section 4.5 and $x=(k^2/k,z^1,z^2)$ in Section 4.6. The heterogeneous agent problems can be summarized using experts' and households' HJB equations, as well as two functional equations for χ and κ that are contained in Appendix B,[8]

$$\begin{aligned}
L^e_{HJB}\left(v^e,v^h,\kappa,\chi;x\right) &= \frac{\rho_e}{1-\rho_e}\delta_e^{1/\rho_e}\exp\left[(1-\frac{1}{\rho_e})v^e)\right]-\frac{\delta_e}{1-\rho_e}+r \\
&+\frac{1}{2\gamma_e}\frac{\left(\Delta^e+\pi^h\cdot\sigma_R\right)^2}{\|\sigma_R\|^2}+\left[\mu_X+\frac{1-\gamma_e}{\gamma_e}\left(\frac{\Delta^e+\pi^h\cdot\sigma_R}{\|\sigma_R\|^2}\right)\sigma_X\sigma_R\right]\cdot\partial_X v^e \\
&+\frac{1}{2}\left[\operatorname{tr}\left(\sigma_X'\partial_{xx'}v^e\sigma_X\right)+\frac{1-\gamma_e}{\gamma_e}\left(\sigma_X'\partial_x v_e\right)'\left[\gamma_e\mathbb{I}_d+(1-\gamma_e)\frac{\sigma_R\sigma_R'}{\|\sigma_R\|^2}\right]\sigma_X'\partial_x v^e\right]=0
\end{aligned} \tag{56}$$

$$\begin{aligned}
L^h_{HJB}\left(v^e,v^h,\kappa,\chi;x\right) &= \frac{\rho_h}{1-\rho_h}\delta_h^{1/\rho_h}\exp\left[(1-\frac{1}{\rho_h})v^h)\right]-\frac{\delta_h}{1-\rho_h}+r+\frac{1}{2\gamma_h}\|\pi^h\|^2 \\
&+\left[\mu_X+\frac{1-\gamma_h}{\gamma_h}\sigma_X\pi^h\right]\cdot\partial_x v^h+\frac{1}{2}\left[\operatorname{tr}\left(\sigma_X'\partial_{xx'}v^h\sigma_X\right)+\frac{1-\gamma_h}{\gamma_h}\left\|\sigma_X'\partial_x v^h\right\|^2\right]=0
\end{aligned} \tag{57}$$

$$L_\kappa\left(v^e,v^h,\kappa,\chi;x\right)=0 \tag{58}$$

$$L_\chi\left(v^e,v^h,\kappa,\chi;x\right)=0 \tag{59}$$

where $x=(w,z^1,z^2)$.

The PDEs for v in equations (21) and v^e,v^h in equation (56) and (57) have the general

[7] Thanks to Judy Yue and Suri Chen for their helpful comments on improving the usability of the computational code.

[8] The experts and households HJB equations for the heterogeneous-agent models in Section 5 share the general form in equation (1). Additionally, L_κ can be formulated using equation (48) and L_χ can be formulated using equation (38).

quasi-linear form

$$0 = A^* \left(x, v, \partial_x v\right) + \mu_X \left(x, v, \partial_x v\right) \partial_x v + \operatorname{tr}\left[B\left(x, v, \partial_x v\right) \partial_{xx'} v B\left(x, \phi_k, \partial_x v\right)'\right] \tag{60}$$

Note that we are able to achieve this form by substituting the minimizing drift adjustment H_t^* into the HJB equation using equation (10) so that the term becomes part of $A^*(x, v, \partial_x v)$ in (60).

To solve, we augment (60) with a false time-derivative $\partial_t v$, known as a "false transient". Since the time-derivative appears on the right-hand-side of the PDE, the equation to solve is

$$0 = \partial_t v + A^* \left(x, v, \partial_x v\right) + \mu_X \left(x, v, \partial_x v\right) \partial_x v + \operatorname{tr}\left[B\left(x, v, \partial_x v\right) \partial_{xx'} v B\left(x, \phi_k, \partial_x v\right)'\right] \tag{61}$$

Thus, the original PDE (60) is the stationary solution to the augmented PDE (61), i.e., $\partial_t v = 0$ holds in (61). We solve (61) iteratively until $\partial_t v \approx 0$.

We break down our approach into three algorithms. Algorithm 1 updates the value function given a generic set of equilibrium objects $\hat{c}, \hat{\imath}, \hat{\kappa}$ etc. - this is called the "outer loop". Algorithm 2 describes how we compute the equilibrium objects for the single-agent models while Algorithm 3 describes the same for environment IP in section 5 (the "inner loop").

Algorithm 1 Finite difference methods for PDEs

1: Form a guess for $\phi_0(x) := v(x, T)$, which is the terminal condition.
2: Generate a grid of time points $\{T, T - \Delta t, \ldots\}$ and a grid of space points $\mathcal{X}$.
3: Given a candidate function $\phi_k(x)$ for $v(x, T - k\Delta t)$ restricted to $\mathcal{X}$, compute finite difference approximations to all derivatives. The time derivative is approximated with the backward difference

$$\partial_t v(x, T - k\Delta t) \approx \frac{v(x, T - k\Delta t) - v(x, T - (k+1)\Delta t)}{\Delta t} = \frac{\phi_k(x) - \phi_{k+1}(x)}{\Delta t}$$

Denote the finite-difference approximations of the spatial derivatives, i.e. the derivatives of the value function with respect to the state variables, by

$$\hat{\partial}_x \phi_k(x) \approx \partial_x v(x, T - k\Delta t)$$
$$\hat{\partial}_{xx'} \phi_k(x) \approx \partial_{xx'} v(x, T - k\Delta t).$$

We apply these approximations to (61) to solve for ϕ_{k+1} given ϕ_k, using one of the schemes in (62) and (63).
4: Using ϕ_{k+1}, calculate

$$\text{error}_{k+1} := \max_{x \in \mathcal{X}} \frac{|v(x, T - k\Delta t) - v(x, T - (k+1)\Delta t)|}{\Delta t} = \max_{x \in \mathcal{X}} \frac{|\phi_{k+1}(x) - \phi_k(x)|}{\Delta t}.$$

Given a tolerance for convergence tol > 0, repeat this step until $\text{error}_{k+1} < \text{tol}$. The function $\phi_{k+1}(x)$ is the approximate solution to (60).

There are several considerations when applying Algorithm 1:

Discrete grid: We use $\Delta_t = 1.0$ when possible to reduce time to convergence, but reduce Δ_t to 0.01 and 0.001 in two cases (Model IP and the two capital model with $\tau = 1$) where we would otherwise experience problems in convergence. For the models where we use $\Delta_t = 1.0$, we find that lowering Δ_t does not alter our converged solutions. We also experiment with different sizes and densities for our state space grids and find no difference in the results (other than computational time).

Approximation of spatial derivatives: We can find ϕ_{k+1} explicitly or implicitly and assume ϕ_k is known. The explicit method approximates the spatial derivatives in (61) using ϕ_k, so that ϕ_{k+1} only appears on the left-hand side of the equation:

$$\begin{aligned} \phi_{k+1} = &\phi_k + \Big\{ A^* \left(x, v, \hat{\partial}_x \phi_k\right) + \mu_X \left(x, \phi_k, \hat{\partial}_x \phi_k\right) \hat{\partial}_x \phi_k \\ &+ \operatorname{tr} \left[B \left(x, \phi_k, \hat{\partial}_x \phi_k\right) \hat{\partial}_{xx'} \phi_k B \left(x, \phi_k, \hat{\partial}_x \phi_k\right)' \right] \Big\} \Delta t. \end{aligned} \tag{62}$$

Notice the right-hand side of (62) can be written as a matrix-vector product (recall that ϕ_k and its partial derivatives are known).

The implicit scheme solves for ϕ_{k+1} in (63). Notice how ϕ_{k+1} (rather than ϕ_k) appears in (63):

$$\begin{aligned}\phi_{k+1} - \phi_k =& \Big\{ A^* \left(x, v, \hat{\partial}_x \phi_k\right) + \mu_X \left(x, \phi_k, \hat{\partial}_x \phi_k\right) \hat{\partial}_x \phi_{k+1} \\ &+ \operatorname{tr}\left[B\left(x, \phi_k, \hat{\partial}_x \phi_k\right) \hat{\partial}_{xx'} \phi_{k+1} B\left(x, \phi_k, \hat{\partial}_x \phi_k\right)' \right] \Big\} \Delta t.\end{aligned} \tag{63}$$

Recall that ϕ_{k+1} and its partial derivatives are unknown. (63) is a linear partial differential equation that can be solved using a finite difference method. In summary, the explicit scheme requires us to compute matrix-vector products whereas the implicit scheme requires us solving a linear system.

The advantage of the implicit scheme is that it tends to work robustly even for larger Δ_t, whereas the explicit scheme typically requires a sufficiently small Δ_t (Achdou et al. (2022)). As such, the implicit scheme provides faster convergence and greater numerical stability. Therefore, we opt for the implicit scheme.

Approximation of drift terms: In (62), we approximate the term $\mu_X\left(x, \phi_k, \hat{\partial}_x \phi_k\right) \hat{\partial}_x \phi_k$ using an upwinding scheme. For each state X_i we compute its contribution to the above term as:

$$\begin{aligned}\mu_{X_i}\left(x, \phi_k, \hat{\partial}_x \phi_k\right) \hat{\partial}_x \phi_k =& \max\{\mu_{X_i}, 0\} \frac{v(x_i + \Delta x_i, x_{-i}, T - k\Delta t) - v(x_i, x_{-i}, T - k\Delta t)}{\Delta x_i} \\ &+ \min\{\mu_{X_i}, 0\} \frac{v(x_i, x_{-i}, T - k\Delta t) - v(x_i - \Delta x_i, x_{-i}, T - k\Delta t)}{\Delta x_i} \\ =& \mu^+_{X_i}(x, \phi_k, \hat{\partial}_x \phi_k) \hat{\partial}^{(+)}_x(\phi_k) + \mu^-_{X_i}(x, \phi_k, \hat{\partial}_x \phi_k) \hat{\partial}^{(-)}_x(\phi_k)\end{aligned}$$

where x_i denotes the value of state X_i at x, x_{-i} denotes the values of the rest of the state variables at x, $\hat{\partial}^{(+)}_x$and $\hat{\partial}^{(-)}_x$ are forward and backward differences, while $\mu^+_{X_i}$ and $\mu^-_{X_i}$ denote the positive and negative parts of μ_{X_i}. Barles and Souganidis (1991) show that this upwinding is necessary, with certain additional regularity conditions, for the numerical scheme to converge to the unique viscosity solution of the underlying partial differential equation, though we do not verify whether these conditions have been satisfied here.

Solving the linear system: The linear system in (62) can be expressed as:

$$Av = u$$

where v is the stacked vector $\phi_{k+1}(x)$ at each point in the state-space grid, u is the flow term and A is a sparse square matrix with dimension equal to the product of the dimensions of the state space grid. In the single-agent models, we solve this using Julia's base function for solving linear systems, which uses an LU decomposition and back-substitution to solve the system.

For Model IP, to solve the linear systems repeatedly, we use a conjugate gradient method, an iterative method that efficiently solves large, sparse, symmetric positive definite linear systems by constructing a sequence of orthogonal search directions to minimize the residual and converge to the solution. There are two advantages associated with conjugate gradient for our solution method. First, notice that for a given linear system, conjugate gradient requires that the user provides an initial guess. In our model, as ϕ_k converges to the true solution, one can reasonably suspect that the distance between ϕ_{k+1} and ϕ_k shrinks as ϕ_k converges. We can then use a "smart guess" approach where, when solving for ϕ_{k+1}, we use ϕ_k as the initial guess. Second, when constructing the finite difference matrix, a smaller Δ_t increases the diagonal of the matrix, making the matrix better-conditioned. This is particularly useful when we solve for models that require smaller Δ_t, because the time required to solve each linear system declines as Δ_t drops.

Now we apply Algorithm 1 to the single and heterogeneous agent models:

Algorithm 2 Numerical Procedure for Single Agent Models using FDM

Given $v^{(n)}$, we would like to update $v^{(n+1)}$ by iterating one time-step in its PDE.

1: **Inner loop: update equilibrium objects iteratively.** For any equilibrium object y, let the sequence of iterants for this loop be $\left\{\hat{y}^{(l)} : l = 0, 1, \dots\right\}$. Form some initial guess for $v^{(0)}$. At the n^{th} step in the iteration process:

1. Solve for the consumption and investment policy functions $\hat{c}^{(n)}$, $\hat{i}^{(n)}$ by applying $v^{(n)}$ to equations (22), (17). Solve also for the drift distortion term $\frac{1-\gamma}{2} z^2 |\sigma_x' \frac{\partial v}{\partial x}|$ described in (10).
2. When structural ambiguity exists, solve additionally for the robust control variable $\hat{s}^{(n)}$ following Hansen and Sargent (2022).
3. Construct the drift and diffusion terms in (21).

2: **Outer loop: update value function using PDE.** Update the value functions $v^{(n+1)}$ using **Algorithm 1**

Algorithm 3 Numerical Procedure for Heterogeneous Agent Models using FDM

Given $v^{e,(n)}$ and $v^{h,(n)}$, we would like to update $v^{e,(n+1)}$ and $v^{h,(n+1)}$ by iterating one time-step in their PDEs.

1: **Inner loop: update equilibrium objects iteratively.**

For any equilibrium object y, let the sequence of iterants for this inner loop be $\{\hat{y}^{(l)} : l = 0, 1, \ldots\}$

1. If $n \geqslant 1$, initialize $\hat{y}^{(0)} = y^{(n-1)}$. If $n = 0$, use the guess $\hat{\kappa}^{(0)} = w$, $\hat{\chi}^{(0)} = 1$, $\hat{q}^{(0)}$ from equation (14), $\hat{\Delta}^{h,(0)} = 0$, and $\hat{\Delta}^{e,(0)} = \underline{\chi}^{-1}[\hat{\Delta}^{h,(0)} + \frac{a_e - a_h}{\hat{q}^{(0)}}]$ from equation (5).
2. For each $l \geqslant 0$, do the following:
   (a) Update all other $\hat{y}^{(l)}$ objects as follows.
      i. Set $\hat{\beta}_e^{(l)} = \hat{\chi}^{(l)}\hat{\kappa}^{(l)}/w$ and $\hat{\beta}_h^{(l)} = (1 - \hat{\kappa}^{(l)})/(1 - w)$.
      ii. Set $\hat{\sigma}_K^{(l)}$, $\hat{\sigma}_q^{(l)}$, $\hat{\sigma}_R^{(l)}$, and $\hat{\pi}^{(l)}$ (in that order) using equations (18), (28), (29), and (33).
      iii. Set $\hat{\mu}_K^{(l)}$, $\hat{\mu}_q^{(l)}$, and $\hat{r}^{(l)}$ (in that order) using equations (18), (26), and (32). Get $\hat{\mu}_R^{e,(l)}$ and $\hat{\mu}_R^{h,(l)}$ from equations (2) and (3), respectively..

   (b) Define $\hat{\kappa}^{(l+1)} = \hat{\kappa}^{(l)} + H^{(l)} \times dt$, where dt is a small enough time-step, and $H^{(l)}$ is defined by the right-hand-side of equation (48), computed using $\chi = \underline{\chi}$ and $\hat{y}^{(l)}$ for all other objects.

   (c) Denote the linear expression in the second argument of the minimum in equation (38) by
   $$G(w, \chi) := A_0(w) + A_1(w)(\chi - w)$$
   Define $\tilde{q}$ according to equation (14) with $\kappa = 1$. Using $\tilde{q}$ and its derivatives in place of q, as well as $\kappa = 1$ and $\hat{y}^{(l)}$, compute A_0, A_1. Solve the equation $G(w, \chi) = 0$ for χ at each w. Denote the solution by $\tilde{\chi}$. If $\tilde{\chi} \geqslant \underline{\chi}$, set $\hat{\chi}^{(l+1)} = \tilde{\chi}$. Otherwise, there are two cases:
   - If $G(w, \underline{\chi}) > 0$, then set $\hat{\chi}^{(l+1)} = \underline{\chi}$.
   - If $G(w, \underline{\chi}) < 0$, then set $\hat{\chi}^{(l+1)} = +\infty$ (or some very large number).

   (d) Use equation (37) to solve for $\hat{\Delta}^{e,(l+1)}$, then set $\hat{\Delta}^{h,(l+1)}$ by (47). Use $\hat{\kappa}^{(l+1)}$ and $\hat{\chi}^{(l+1)}$ but $\hat{y}^{(l)}$ for everything else in this step.

   (e) Set $\hat{q}^{(l+1)}$ by equation (14), using $\hat{\kappa}^{(l+1)}$ and $(v^{e,(n)}, v^{h,(n)})$.
3. Iterate on (b). When $\|\hat{\kappa}^{(l+1)} - \hat{\kappa}^{(l)}\| + \|\hat{\chi}^{(l+1)} - \hat{\chi}^{(l)}\|$ is small, stop iterating.
4. Put $y^{(n)} = \hat{y}^{(l)}$.

2: **Outer loop: update value functions using PDEs.** Update the experts and households value functions $v^{e,(n+1)}, v^{h,(n+1)}$ separately using **Algorithm 1**

C.2 Approximate global solutions using neural networks

We globally solve the heterogeneous agents models in environments RF, PR, SG in section 5 using machine learning methods. Amidst the backdrop of rapid advancements in deep learning, Sirignano and Spiliopoulos (2018) proposed the deep Galerkin method (DGM) to solve partial differential equations using neural networks without relying on mesh generation. Al-Aradi et al. (2022) handled HJB equations in their original, unsimplified form, solving for the value function and optimal control by representing each with deep neural networks, and proceed with policy iteration algorithm (DGM-PIA). Barnett et al. (2023) systematically review the recent development for deep learning algorithms in scientific computing.

Contrast with the DGM-PIA algorithm used in Al-Aradi et al. (2022), we approximate expert value function, v^e, households value function, v^h, and expert's capital share κ simultaneously using a single neural network with 3-dimensional outputs. The algorithm avoids any iteration among equilibrium variables to achieve efficiency. We explicitly solve χ using (59) due to its embedded linear structure, and rewrite the system to be solved as

$$\begin{aligned} L^e_{HJB}\left(v^e, v^h, \kappa; x\right) &= 0 \\ L^h_{HJB}\left(v^e, v^h, \kappa; x\right) &= 0 \\ L_\kappa\left(v^e, v^h, \kappa; x\right) &= 0 \end{aligned} \tag{64}$$

Our single neural network has 1 input layer, 2 hidden layers, and 1 final output layer. The input layer takes a 3-dimensional vector, (w, z^1, z^2), as inputs. The final layer has 3-dimensional outputs, which are used to approximate v^e, v^h, κ respectively. Each hidden layer has 16 neurons[9]. tanh activation functions are used in each neuron except for the final layer. In the final layer, we choose no activation function for v^e and v^h, and use a sigmoid activation function for κ to constrain the expert's capital share between 0 and 1. Our neural net approximation F can be characterized as

$$[v^e, v^h, \kappa] = F(x; \boldsymbol{\theta}) \tag{65}$$

where $\boldsymbol{\theta}$ are neural net parameters.

[9] Our results are robust across various neural network architectures. Increasing the number of hidden layers and units do not impact our results.

Algorithm 4 Training Procedure

We use a Glorot normal initializer[1] to initialize our neural nets parameters $\boldsymbol{\theta}_0$. Then given $\boldsymbol{\theta}_i$, $i = 0, 1, 2, ..n$, at each iteration we

1: Uniformly draw (w, z^1, z^2) from the state space N times. The formulated training set $\boldsymbol{x_i} = (\boldsymbol{w_i}, \boldsymbol{z_i^1}, \boldsymbol{z_i^2})$ is a $N \times 3$ matrix.[2]

2: Evaluate the neural nets on the training set

$$[\boldsymbol{v_i^e}, \boldsymbol{v_i^h}, \boldsymbol{\kappa_i}] = F(\boldsymbol{x_i}; \boldsymbol{\theta}_i)$$

and calculate the loss functions

$$\begin{aligned} &\boldsymbol{L}^e_{HJB}\left(\boldsymbol{v_i^e}, \boldsymbol{v_i^h}, \boldsymbol{\kappa_i}; \boldsymbol{x_i}, \boldsymbol{\theta}_i\right) \\ &\boldsymbol{L}^h_{HJB}\left(\boldsymbol{v_i^e}, \boldsymbol{v_i^h}, \boldsymbol{\kappa_i}; \boldsymbol{x_i}, \boldsymbol{\theta}_i\right) \\ &\boldsymbol{L}_{\kappa}\left(\boldsymbol{v_i^e}, \boldsymbol{v_i^h}, \boldsymbol{\kappa_i}; \boldsymbol{x_i}, \boldsymbol{\theta}_i\right) \end{aligned} \tag{66}$$

3: Calculate the mean square error for each loss function

$$\begin{aligned} \mathcal{L}^e_{HJB}\left(\boldsymbol{x_i}, \boldsymbol{\theta}_i\right) &= \frac{1}{N}\|\boldsymbol{L}^e_{HJB}\left(\boldsymbol{v_i^e}, \boldsymbol{v_i^h}, \boldsymbol{\kappa_i}; \boldsymbol{x_i}, \boldsymbol{\theta}_i\right)\|^2 \\ \mathcal{L}^h_{HJB}\left(\boldsymbol{x_i}, \boldsymbol{\theta}_i\right) &= \frac{1}{N}\|\boldsymbol{L}^h_{HJB}\left(\boldsymbol{v_i^e}, \boldsymbol{v_i^h}, \boldsymbol{\kappa_i}; \boldsymbol{x_i}, \boldsymbol{\theta}_i\right)\|^2 \\ \mathcal{L}_{\kappa}\left(\boldsymbol{x_i}, \boldsymbol{\theta}_i\right) &= \frac{1}{N}\|\boldsymbol{L}_{\kappa}\left(\boldsymbol{v_i^e}, \boldsymbol{v_i^h}, \boldsymbol{\kappa_i}; \boldsymbol{x_i}, \boldsymbol{\theta}_i\right)\|^2 \end{aligned} \tag{67}$$

where $\mathcal{L}^e_{HJB}, \mathcal{L}^h_{HJB}, \mathcal{L}_{\kappa}$ are all scalars.

4: Construct the composite objective loss function as

$$\mathcal{L} = \mathcal{L}^e_{HJB} + \mathcal{L}^h_{HJB} + \lambda_{\mathcal{L}}\mathcal{L}_{\kappa}$$

where $\lambda_{\mathcal{L}}$ is a weighting coefficient.[3]

5: Update $\boldsymbol{\theta}_i$ using the standard scipy Broyden–Fletcher–Goldfarb–Shanno optimization algorithm[4] in each iteration until $\boldsymbol{\theta}_n$ and $\mathcal{L}$ fall below the tolerance.[5]

$$\boldsymbol{\theta}_{i+1} = \arg\min \mathcal{L} \tag{68}$$

C.3 Computing Shock Elasticities

The shock elasticities (exposure and price elasticities) are computed by solving:

$$\varepsilon_M(t,x) = \nu(x) \cdot \Big\{\sigma_M(x) + \sigma_X(x) \cdot \frac{\partial}{\partial x} \log \mathbb{E}\Big[\Big(\frac{M_t}{M_0}\Big) \mid X_0 = x\Big]\Big\} \tag{69}$$

where μ_M, σ_M are the drift and diffusion of $d \log M_t$. See Borovička et al. (2014) and Borovička and Hansen (2016) for further discussion. We compute this in two ways: 1) by solving a PDE using finite differences, or 2) using simulations.

First, we can compute the conditional expectation in (69) using finite differences as follows. Define $f_M(t,x) := \mathbb{E}[\frac{M_t}{M_0} f_M(0, X_t) \mid X_0 = x]$. Then, using the law of iterated expectations, followed by the definition of f_M, we have $f_M(t,x) = \mathbb{E}[\frac{M_u}{M_0}\mathbb{E}[\frac{M_t}{M_u} f_M(0,X_t) \mid X_u] \mid X_0 = x] = \mathbb{E}[\frac{M_u}{M_0} f_M(t-u, X_u) \mid X_0 = x]$. Hence, $\{M_t f_M(T-t, X_t)\}_{t\in[0,T]}$ is a martingale and must have zero drift. Applying Itô's formula gives a PDE for f_M in (t,x), i.e.,

$$0 = -\frac{\partial f_M}{\partial t} + \Big(\mu_M + \frac{1}{2}\|\sigma_M\|^2\Big) f_M + \big(\mu_X + \sigma_M \cdot \sigma_X\big) \cdot \frac{\partial f_M}{\partial x} + \frac{1}{2}\mathrm{tr}\Big(\sigma_X \sigma_X' \frac{\partial^2 f_M}{\partial x \partial x'}\Big). \tag{70}$$

The initial condition is $f_M(0,x) \equiv 1$, which allows us to recover the desired conditional expectation. The linear PDE in (70) is solved using finite difference methods. We obtain $\varepsilon_M(t,x)$ by numerically differentiating $f_M(t,x)$ and substituting it into (69).

Alternatively, we can compute the expectation term by simulating M using μ_M and σ_M and taking the mean across simulations for each (t,x). We find that both methods generate the same results for the figures used in the paper. Discrepancies, however, arise when we initialize the state variable near the boundaries of the state space grid for some example economies not reported in the paper.

The left panel of Figure 1 shows discrepancies between the simulation method (which is the accurate solution) and the PDE method due to subtle issues related to local martingales embedded in M. For instance, this figure is based on environment RF, which features a

[1] Further details on TensorFlow's Glorot normal initializer are available here.

[2] Our results are robust to increasing the number of points in each individual training set.

[3] In our training, we prioritize ensuring that first-order conditions are met by setting $\lambda_{\mathcal{L}} = 10{,}000$. This penalization effectively reduces L_κ to a range between 10^{-11} and 10^{-14}.

[4] A detailed explanation of the BFGS algorithm implemented in SciPy can be found here. We choose the BFGS algorithm over Adam due to its higher efficiency in our tests. BFGS achieves a rapid reduction in composite loss and attains lower validation errors across three models.

[5] We choose $n = 5$ and set tolerance as 10^{-4}. Our results are robust for $n \geqslant 5$.

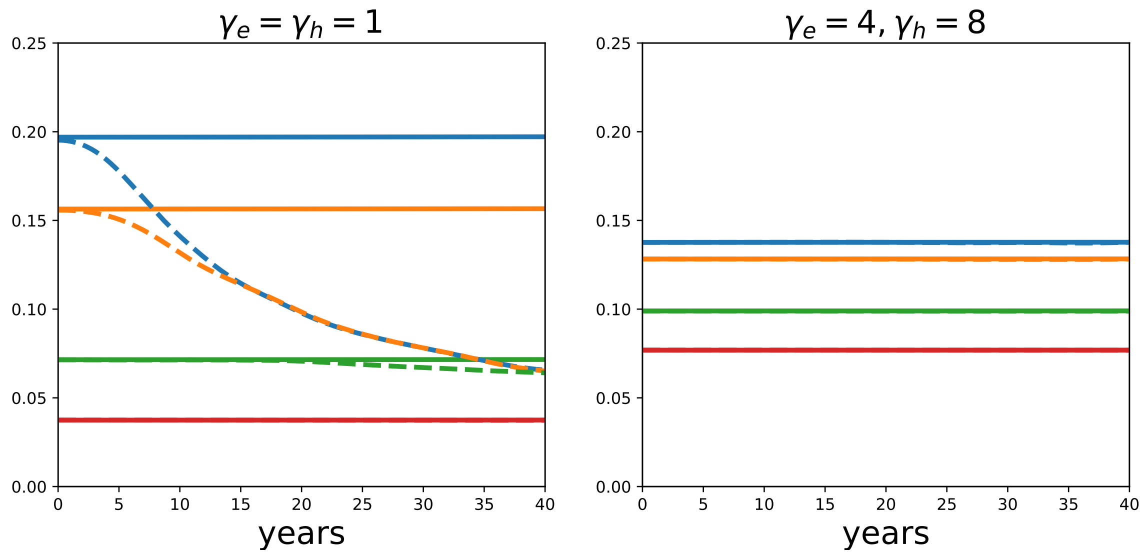

Figure 1: Capital shock price elasticities for environment RF of Section 5. The left plot imposes $\gamma_e = \gamma_h = 1$ while the right plot imposes $\gamma_e = 4$ and $\gamma_h = 8$; the remaining parameters are the same as in Section 5. The solid lines represent elasticities computed using finite differences, while the dashed lines represent elasticities computed using simulations. The blue curve gives elasticities when W is initialized at the 0.5th percentile of its stationary distribution, the orange curve at the 1st percentile, the green curve at the 10th percentile and the red curve at the median. For the first parameterization, the difference between the two methods becomes significant as the initial point moves closer to the boundary at zero, whereas the two methods coincide in the second parameterization. The simulated elasticities were computed using 5000 Monte Carlo simulation draws for time trajectories of forty years. We drop all simulations where, at some time during the simulation, wealth is less than the smallest grid point (0.001). We drop 0.08 percent of simulations for the elasticities initialized at the 0.5th percentile, and less for the other initializations. The simulated elasticities are smoothed using a Gaussian kernel with standard deviation of 200, where the time increment is 0.01 years.

singularity at $w = 0$; namely, since experts cannot deleverage, their risk prices diverge to $+\infty$ as $w \to 0$. This singularity can be so severe as to render the risk-neutral probability density a strict local martingale, rather than a true martingale. See Hugonnier (2012) for a theoretical analysis of this situation in a restricted participation model, including the possibility of bubbles and the like. When such local martingales arise, computing shock elasticities using the PDE method requires delicate consideration of boundary conditions in the space dimension (x). For instance, in environment RF with log preferences ($\gamma_e = \gamma_h = 1$ as in the left panel), we can prove that both "natural boundary conditions" and "reflecting boundary conditions" will lead to the wrong answers for the elasticities (on the other hand, the simulations will remain correct).

In principle, these local martingale issues could cause numerical challenges for environments RF and SG, both of which feature this local martingale issue. That being said, the right panel shows that the preference parameters used in the paper (in particular the large level of risk aversions) strongly mitigate the local martingale issues. Hence, we do not expect substantial issues for any calculations in the paper. Furthermore, we have always double-checked any elasticity calculations using both the PDE method as well as a simulation-based approach.

The "term structure of uncertainty prices" is formed via a second type of shock elasticity, which differs conceptually from the first type described above. While $\varepsilon_M(t,x)$ measures the expected response of M_t to a shock at time 0, we could also compute the expected response of M_t to a shock at the same time t. We can compute this alternative shock elasticity via

$$\tilde{\varepsilon}_M(t,x) = \nu(x) \cdot \frac{\mathbb{E}[\frac{M_t}{M_0}\sigma_M(X_t) \mid X_0 = x]}{\mathbb{E}[\frac{M_t}{M_0} \mid X_0 = x]}, \tag{71}$$

The calculation of term structure of uncertainty prices requires solving the PDE (70) with initial conditions $f_M(0,x) \equiv \sigma_M(x)$ to obtain the numerator. Once again, this conditional expectation can alternatively be solved using simulations. Note that, for the uncertainty prices computed the paper, the process M will be a martingale representing uncertainty-induced belief distortions; consequently, the denominator is 1 due to the martingale property.

D Computational Methods by Figure

Figures	Method of Computation
1, 2 and 3	Using a two-dimensional finite difference method, as detailed in Appendix C.1, we first solve the model for $\rho = 1$ and use the solution as an initial guess for $\rho = 0.67$ and 1.5. We then compute shock elasticities using finite differences as outlined in Appendix C.3.
4	We follow the method described in Hansen and Sargent (2022) Section 4.2.2. For a given level of relative entropy q, we use the method of undetermined coefficients to solve for set of the drift distortion parameters (η_0, η_1) satisfying the restriction on ρ. For each set of (η_0, η_1), we then compute the corresponding structural parameters $\beta_1, \beta_k, \eta_k, \phi_1$ and draw the contour set.
5 and 6	Using a one-dimensional finite difference method, we solve the baseline model, the worst-case structured model and compute shock elasticities.
7, 8 and 9	For computational reasons, we first solve a version of the model without stochastic volatility using a two-dimensional finite difference method. We then use the solutions as initial guesses for the corresponding specifications in the model with stochastic volatility and solve using a three-dimensional finite difference method.
10, 11 and 12	We solve each model using neural networks as detailed in Appendix C.2.
13	We solve the model using a two-dimensional finite differences method, via the MFR Suite library.
14 and 15	We use finite differences methods to compute the shock elasticities as outlined in Appendix C.3.

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