

Multiplicative Functionals

Contents

- 7.1. Geometric growth and decay
- 7.2. Special multiplicative functionals
- 7.3. Multiplicative martingales
- 7.4. Factoring a multiplicative functional
- 7.5. Stochastic stability
- 7.6. Inferences about permanent shocks
- 7.7. Observable counterparts to a factorization of stochastic discount factors
- 7.8. Long-term risk-return tradeoff for cash flows
- 7.9. Bounding investor beliefs

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Challenges in navigating long-term uncertainty.

[Chapter 4: Processes with Markovian increments](#) described **additive functionals** of a Markov process. This chapter describes exponentials of additive functionals that we call **multiplicative functionals**. We can use them to model stochastic growth, stochastic discounting, belief distortions and their interactions. After adjusting for geometric growth or decay, a multiplicative functional contains a martingale component that turns out to be a likelihood ratio process that is itself a special type of multiplicative functional called an **exponential martingale**. By simply multiplying random variables of interest by the multiplicative martingale prior to computing conditional expectations under a baseline probability, we construct an alternative probability measure. It functions as a relative density that alters the baseline probability measure. For an initial application of multiplicative functionals to asset valuation and investor preferences, see [[Anderson et al., 2003](#)]. We will explore these and other applications in discussions that follow. We will see that multiplicative functionals has several applications. They allow us to characterize components of stochastic growth and stochastic discounting that persist over long time horizons. They show how macroeconomic shocks with long-term impacts are reflected in asset pricing. They provide a tractable modeling tool for models in which investors have subjective beliefs that may deviate from a baseline probability and allow for measuring their magnitudes using statistical discrimination measures. We will encounter several other applications of multiplicative functionals, including models of returns and positive cash flows that compound over multiple horizons, cumulative stochastic discount factors are used to long-horizon, risk-return tradeoffs.

7.1. Geometric growth and decay

To construct a multiplicative functional, we start with an underlying Markov process X that has a stationary distribution Q , and we suppose that Y_0 and X_0 depend only on date zero information summarized by $\mathfrak{A}_0$.

Definition 7.1

Let $Y \stackrel{\text{def}}{=} \{Y_t\}$ be an additive functional that as in [Chapter 4](#) is described by

$$Y_{t+1} - Y_t = \kappa(X_t, W_{t+1}),$$

where X_t is the time t component of a Markov state vector satisfying $X_{t+1} = \phi(X_t, W_{t+1})$ and W_{t+1} is the time $t + 1$ value of a martingale difference process ($\mathbb{E}(W_{t+1} \mid \mathfrak{A}_t) = 0$) of unanticipated shocks. We say that $M \stackrel{\text{def}}{=} \{M_t : t \geq 0\} = \{\exp(Y_t) : t \geq 0\}$ is a **multiplicative functional** parameterized by κ .

An additive functional grows or decays *linearly*, so the exponential of an additive functional grows or decays *geometrically*. We construct a multiplicative functional recursively by

$$M_{t+1} = N_{t+1}M_t. \tag{7.1}$$

We may solve for N_{t+1} ,

$$N_{t+1} = \begin{cases} \frac{M_{t+1}}{M_t}, & M_t > 0 \\ 1, & M_t = 0 \end{cases}$$

When M_t is zero, so is M_{t+1} . In this case, the random variable, N_{t+1} , is not uniquely defined by [\(7.1\)](#). Our decision to set it to unity is simply a convenient normalization. In light of [\(7.1\)](#), we refer to N as the **multiplicative increment** of the process M .

[Chapter 4](#) stated a Law of Large Numbers and a Central Limit Theorem for additive functionals. In this chapter, we use other mathematical tools to analyze the limiting behavior of multiplicative functionals.

7.2. Special multiplicative functionals

We define the three primitive multiplicative functionals.

Example 7.1

Suppose that $\kappa = \eta$ is constant and that M_0 is a Borel measurable function of X_0 . Then

$$M_t = \exp(t\eta)M_0.$$

This process grows or decays geometrically.

Example 7.2

Suppose that

$$\mathbb{E}[\exp[\kappa(X_t, W_{t+1})] | \mathfrak{A}_t] = 1.$$

Then

$$\mathbb{E}(M_{t+1} | \mathfrak{A}_t) = M_t \tag{7.2}$$

so that

$$\mathbb{E}(N_{t+1} | \mathfrak{A}_t) = 1$$

A multiplicative functional that satisfies [\(7.2\)](#) is called a *multiplicative martingale*. We denote such a process as $M \stackrel{\text{def}}{=} L$ because it is appropriate to view it as likelihood ratio process. We will have more to say about this in some of the discussion that follows. It will often be convenient to initialize this process at $M_0 = L_0 = 1$.

Example 7.3

Suppose that $M_t = \exp[h(X_t)]$ where h is a Borel measurable function. The associated additive functional satisfies

$$\begin{aligned} Y_{t+1} - Y_t &= \log M_{t+1} - \log M_t \\ &= h(X_{t+1}) - h(X_t) \\ &= h[\phi(X_t, W_{t+1})] - h(X_t) \end{aligned}$$

and is parameterized by $\kappa(X_t, W_{t+1}) = h[\phi(X_t, W_{t+1})] - h(X_t)$ with initial condition $Y_0 = h(X_0)$.

When the process $\{X_t\}$ is stationary and ergodic, multiplicative functional [Example 7.1](#) displays expected growth or decay, while multiplicative functionals [Example 7.2](#) and [Example 7.3](#) do not. Multiplicative functional [Example 7.3](#) is stationary, while [Example 7.1](#) and [Example 7.2](#) are not.

We can construct other multiplicative functionals simply by multiplying instances of these primitive ones. Soon we shall reverse that process by taking an arbitrary multiplicative functional and (multiplicatively) decomposing it into instances of our three types of multiplicative functionals. Before doing so, we explore multiplicative martingales in more depth.

7.3. Multiplicative martingales

We can use multiplicative martingales to represent alternative probability models. We can characterize an alternative model with a set of implied conditional expectations of all bounded random variables, B_{t+1} , that are measurable with respect to $\mathfrak{A}_{t+1}$. The constructed conditional expectation is

$$\mathbb{E}(N_{t+1}B_{t+1} \mid \mathfrak{A}_t). \quad (7.3)$$

We want multiplication of B_{t+1} by N_{t+1} to change the baseline probability to an alternative probability model. To accomplish this, the random variable N_{t+1} must satisfy:

1. $N_{t+1} \geq 0$;
2. $\mathbb{E}(N_{t+1} \mid \mathfrak{A}_t) = 1$;
3. N_{t+1} is $\mathfrak{A}_{t+1}$ measurable.

Property 1 is satisfied because conditional expectations map positive random variables B_{t+1} into positive random variables that are $\mathfrak{A}_t$ measurable. Properties 2 and 3 are satisfied because N is the multiplicative increment of a multiplicative martingale. The resulting process L can be viewed as a likelihood ratio or Radon-Nikodym derivative process for the alternative probability measure relative to the baseline measure.

Representing an alternative probability model in this way is restrictive. For instance, if a nonnegative random variable has conditional expectation zero under the baseline probability, it will also have zero conditional expectation under the alternative probability measure, an indication of *absolute continuity* of the implied probability measure with respect to the baseline measure. Two models that violate absolute continuity can be distinguished with probability one from only finite samples. To avoid this degenerate outcome, likelihood-based statistical inference typically imposes this form of absolute continuity.

Given the multiplicative construction, the implied alternative conditional probability measure over τ periods uses the random variable

$$\prod_{j=1}^{\tau} N_{t+j}$$

to compute the τ time-period-ahead conditional expectation. With this construction, the τ -period conditional expectation may be computed by iterating on the one-period conditional expectations in

accordance with the Law of Iterated expectations.

Multiplicative martingales provide a way to model diverse subjective beliefs of private agents or policy-makers within dynamic, stochastic equilibrium models when these beliefs are allowed to depart from the model builder's model. As [\[Hansen and Scheinkman, 2009\]](#) show, they also offer a way to value cumulative returns. Let R_t be a multiplicative process that measures a cumulative return between date t and date zero. Let S_t be a corresponding equilibrium discount factor between these same two dates. That $L = RS$ is a multiplicative martingale follows from equilibrium restrictions on one-period returns. That is,

$$\mathbb{E} \left[\left(\frac{S_{t+1}}{S_t} \right) \left(\frac{R_{t+1}}{R_t} \right) \mid \mathfrak{A}_t \right] = 1,$$

where S_{t+1}/S_t is the one-period stochastic discount factor and R_{t+1}/R_t is the one-period gross return. For the application, we construct

$$N_{t+1} = \frac{S_{t+1}R_{t+1}}{S_tR_t}$$

Here are some examples of multiplicative martingales constructed from some standard probability models.

Example 7.4

Consider a baseline Markov process having transition probability density π_o with respect to a measure λ over the state space $\mathcal{X}$

$$P_o(dx^+ | x) = \pi_o(x^+ | x)\lambda(dx^+)$$

Let π denote some other transition density that we represent as

$$\pi(x^+ | x)\lambda(dx^+) = \left[\frac{\pi(x^+ | x)}{\pi_o(x^+ | x)} \right] \pi_o(x^+ | x)\lambda(dx^+)$$

where we assume that $\pi_o(x^+ | x) = 0$ implies that $\pi(x^+ | x) = 0$ for all x^+ and x in $\mathcal{X}$. Construct the multiplicative increment process as:

$$N_{t+1} = \frac{\pi(X_{t+1} | X_t)}{\pi_o(X_{t+1} | X_t)}.$$

Example 7.5

Let an alternative model for a vector X be a vector autoregression:

$$X_{t+1} = \mathbb{A}X_t + \mathbb{B}W_{t+1}$$

where $\mathbb{A}$ is a stable matrix, $\{W_{t+1} : t \geq 0\}$ is an i.i.d. sequence of $\mathcal{N}(0, I)$ random vectors conditioned on X_0 , and $\mathbb{B}$ is a square, nonsingular matrix. Assume that a baseline model for X has the same functional form but different settings $(\mathbb{A}_o, \mathbb{B}_o)$ of its parameters. Construct N_{t+1} as the one-period conditional log-likelihood ratio

$$\begin{aligned} \log N_{t+1} = & -\frac{1}{2}(X_{t+1} - \mathbb{A}X_t)'(\mathbb{B}\mathbb{B}')^{-1}(X_{t+1} - \mathbb{A}X_t) \\ & + \frac{1}{2}(X_{t+1} - \mathbb{A}_oX_t)'(\mathbb{B}_o\mathbb{B}_o')^{-1}(X_{t+1} - \mathbb{A}_oX_t) \\ & - \frac{1}{2}\log \det(\mathbb{B}\mathbb{B}') + \frac{1}{2}\log \det(\mathbb{B}_o\mathbb{B}_o') \end{aligned}$$

Notice how the matrices $(\mathbb{A}_o, \mathbb{B}_o)$ of the baseline model and parameters $(\mathbb{A}, \mathbb{B})$ of the alternative model both appear.

Remark 7.1

Because $\mathbb{B}$ is a nonsingular square matrix, model [Example 7.5](#) has the same number of shocks, i.e., entries of W , as there are components of X . A more general setting would be a hidden Markov state model like one presented in Section [Kalman Filter and Smoother](#) of Chapter [Hidden Markov Models](#) that has a time-invariant representation with an “information state vector” constructed as a way to condition on an infinite past of an observation vector.

We can elicit a limiting behavior of multiplicative martingales by applying Jensen’s inequality to the concave function $\log L$ depicted in [Fig. 7.1](#) based on

$$\mathbb{E}(\log L_t \mid \mathfrak{A}_0) \leq 0$$

where we normalize $L_0 = 1$.

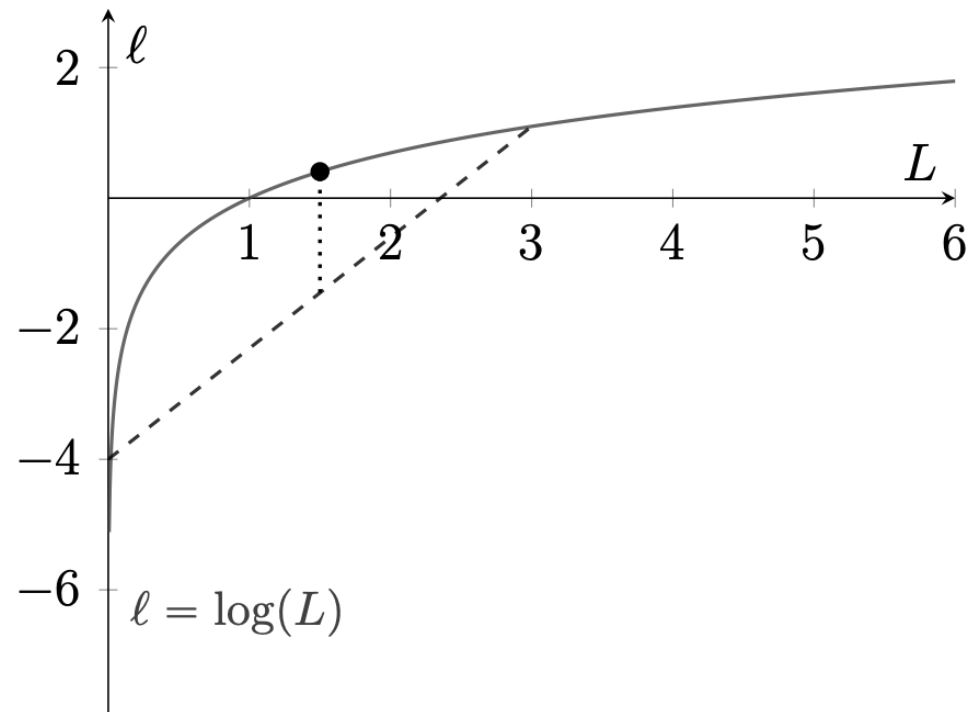


Fig. 7.1 Jensen's Inequality. The logarithmic function is a concave function that equals zero when evaluated at unity. The line segment lies below the logarithmic function. An interior average of endpoints of the straight line lies below the logarithmic function.

Moreover, by Jensen's inequality,

$$\mathbb{E}(\log N_{t+1} \mid \mathfrak{A}_t) \leq \log \mathbb{E}(N_{t+1} \mid \mathfrak{A}_t) = 0.$$

for N_{t+1} satisfying $L_{t+1} = N_{t+1}L_t$. Note that

$$\mathbb{E}(\log L_{t+1} \mid \mathfrak{A}_t) = \log L_t + \mathbb{E}(\log N_{t+1} \mid \mathfrak{A}_t) \leq \log L_t.$$

This implies that under the baseline model the log-likelihood ratio process L is a *supermartingale* relative to the information sequence $\{\mathfrak{A}_t : t \geq 0\}$.

From the Law of Large Numbers as described [Chapter 1: Laws of Large Numbers and Stochastic Processes](#), a population mean is well approximated by a sample average from a long time series. That opens the door to discriminating between two models. Under the baseline model, the log likelihood ratio process scaled by $1/t$ converges to a negative number. If the baseline model, actually generates the data, the expected log likelihood ratio constructed with data will (at least eventually) be negative except in the degenerate case in which $N_{t+1} = 1$ with probability one. Such a calculation justifies discriminating between the two models by calculating $\log L_t$ and checking if it is positive or negative. This procedure amounts to an application of the method of maximum likelihood. Sometimes

$$-\mathbb{E}(\log N_{t+1} \mid \mathfrak{A}_t) \geq 0$$

is used as a measure of statistical divergence.

Suppose now that we reverse the roles of the baseline and alternative probability measures.

Definition 7.2

The **conditional relative entropy** of a martingale increment is defined to be:

$$E(N_{t+1} \log N_{t+1} \mid \mathfrak{A}_t) \geq 0.$$

This entity is sometimes referred as Kullback-Leibler divergence. To understand why conditional relative entropy is nonnegative, observe that multiplication of $\log N_{t+1}$ by N_{t+1} changes the conditional probability distribution for which the conditional expectation of $\log N_{t+1}$ is calculated from the baseline model to the alternative model. The function $n \log n$ is convex and equal to zero for $n = 1$. Therefore, Jensen's inequality implies that conditional relative entropy is nonnegative and equal to zero when $N_{t+1} = 1$ conditioned on $\mathfrak{A}_t$.

Notice that

$$\begin{aligned} \mathbb{E}(L_{t+1} \log L_{t+1} \mid \mathfrak{A}_t) &= L_t \mathbb{E}(N_{t+1} \log N_{t+1} \mid \mathfrak{A}_t) + L_t \log L_t \\ &\geq L_t \log L_t. \end{aligned}$$

Thus $L \log L$ is a *submartingale*. The expression

$$\mathbb{E}(L_t \log L_t \mid \mathfrak{A}_0) \geq 0,$$

and is a measure of relative entropy over a t -period horizon. Relative entropy is often used to analyze model misspecifications and also appears in statistical characterizations of "large deviations" for Markov processes and in information theory.

Remark 7.2

Consider a simple version of likelihood-based model identification. Suppose that a decision-maker does not know whether a baseline or alternative model generates the data. Attach a subjective prior probability π_o to the baseline probability model and probability $1 - \pi_o$ on the alternative. Let L be a likelihood ratio process with L_t reflecting information available at date t . Date t posterior probabilities for the baseline and alternative probability models are:

$$\frac{\pi_o}{L_t(1 - \pi_o) + \pi_o} \quad \text{and} \quad \frac{L_t(1 - \pi_o)}{L_t(1 - \pi_o) + \pi_o}.$$

When $\frac{1}{t} \log L_t$ converges to a negative number under the baseline probability, the first probability converges to one. But when $\frac{1}{t} \log L_t$ converges to a positive number under the alternative probability, the second probability converges to one. When the data are generated by the baseline probability model, the Law of Large Numbers implies the former; and when the

data are generated by the alternative probability model, the Law of Large Numbers implies the latter. This shows that model selection based on posterior probabilities will eventually determine which model generated the data, the baseline model or the alternative model. This analysis can be extended to situations in which some other model generates the data.

Remark 7.3

The tools provided here open the door to the economic analysis of asset pricing models in which investors have so-called “subjective” beliefs. These beliefs could deviate from a baseline probability, sometimes motivated via rational expectations. Much of the behavioral finance literature appeals to insights from psychology to motivate such distortions. The tools here can complement such analyses by providing ways to think about the complexity of the environment within which investors reside. In the behavioral finance literature, you will often read about over and under reaction, but over and under reaction relative to what? The data generating process may itself be very difficult for investors to learn about and for econometricians to isolate. So should that be the relevant benchmark or baseline to measure belief distortions? Moreover, presumable so called distortions are more likely to persist when statistical challenges for the investors and econometricians are substantial. Tools from probability and statistics thus are pertinent for the study belief impacts on asset prices.

Remark 7.4

These tools are also pertinent to the study of models in which investors have heterogeneous beliefs. [[Alchian, 1950](#)] and [[Friedman, 1953](#)], among others, have argued that investors with distorted beliefs will eventually be driven out of the market by investors with more accurate perceptions of the future because of the relative success of the latter type of investors. This has been one argument for imposing rational expectations in asset pricing models. [[Kogan et al., 2006](#)] refine this view by breaking a simple link between survival and price impact of the investors with distorted beliefs. [[Borovička, 2020](#)] goes further by characterizing families of investor preferences for which the investors with the distorted beliefs survive in the long run. These latter two contributions feature differences in how investors look at stochastic growth. A continuous-time counterpart to the martingale representation for belief distortions that we describe and characterize in this chapter are featured in both the [[Kogan et al., 2006](#)] and the [[Borovička, 2020](#)] papers.

7.4. Factoring a multiplicative functional

Following [[Hansen and Scheinkman, 2009](#)] and [[Hansen, 2012](#)], we factor a multiplicative functional into three multiplicative components having the primitive types [Example 7.1](#), [Example 7.2](#), [Example 7.3](#).

As in definition [Definition 7.1](#), let Y be an additive functional, and let $M = \exp(Y)$. Apply a one-period operator $\mathbb{M}$ defined by

$$\begin{aligned}\mathbb{M}f(x) &\stackrel{\text{def}}{=} \mathbb{E} [\exp(Y_{t+1} - Y_t) f(X_{t+1}) \mid X_t = x] \\ &= \mathbb{E} \left[\left(\frac{M_{t+1}}{M_t} \right) f(X_{t+1}) \mid X_t = x \right].\end{aligned}\tag{7.4}$$

to bounded Borel measurable functions f of the Markov state. By applying the Law of Iterated expectations, a two-period operator iterates $\mathbb{M}$ twice to obtain:

$$\begin{aligned}\mathbb{M}^2 f(x) &\stackrel{\text{def}}{=} \mathbb{E} [\exp(Y_{t+2} - Y_t) f(X_{t+2}) \mid X_t = x] \\ &= \mathbb{E} \left[\left(\frac{M_{t+2}}{M_t} \right) f(X_{t+2}) \mid X_t = x \right],\end{aligned}\tag{7.5}$$

with corresponding definitions of j -period operators $\mathbb{M}^j$. The family of operators is a special case of what is called a “semi-group.” The domain of the semigroup can typically be extended to a larger family of functions, but this extension depends on further properties of the multiplicative process used to construct it.

We next derive a revealing representation of this semigroup by factoring the multiplicative functional M in an interesting way. We achieve this by applying what is referred to in mathematics as Perron-Frobenius theory based on the following equation:

Eigenvalue-eigenfunction Problem: Solve

$$\mathbb{M}\tilde{e}(x) = \exp(\tilde{\eta})\tilde{e}(x)\tag{7.6}$$

for an eigenvalue $\exp(\tilde{\eta})$ and a positive eigenfunction $\tilde{e}$.

Consistent with Perron-Frobenius theory, call the positive eigenvalue, the **principal eigenvalue**, and the associated positive eigenvector, the** principal eigenfunction**, of the operator $\mathbb{M}$.

Use the principal eigenvalue and eigenvector, and define:

$$\tilde{N}_{t+1} \stackrel{\text{def}}{=} \exp(-\tilde{\eta}) \frac{M_{t+1}\tilde{e}(X_{t+1})}{M_t\tilde{e}(X_t)} = \exp[\tilde{\kappa}(X_t, W_{t+1})],$$

Note that we constructed $\tilde{N}_{t+1}$ so as to have a conditional expectation equal to unity. Using $\tilde{N}$ as a martingale increment process, build

$$\tilde{L}_{t+1} = \tilde{N}_{t+1}\tilde{L}_t, \quad \tilde{L}_0 = 1.$$

Theorem 7.1

Let M be a multiplicative functional. Suppose that the principal eigenvalue-eigenfunction [Problem](#) has a solution with principal eigenfunction $\tilde{e}(X)$. Then the multiplicative functional is the product of three components that are instances of the primitive functionals in examples [Example 7.1](#), [Example 7.2](#), and [Example 7.3](#):

$$\frac{M_t}{M_0} = \exp(\tilde{\eta}t) \tilde{L}_t \left[\frac{\tilde{e}(X_0)}{\tilde{e}(X_t)} \right] \quad (7.7)$$

where $\tilde{L}_t$ is a multiplicative martingale.

The factorization of a multiplicative functional described in [Theorem 7.1](#) is a counterpart to the [Chapter 4 Proposition 4.1](#) decomposition of an additive functional. We used the [Proposition 4.1](#) martingale to identify the permanent component of an additive functional in [Chapter 4](#). In this chapter, we shall use the multiplicative martingale isolated by [Theorem 7.1](#) to represent a change of probability measure, but one that will help us understand long-term risk return tradeoffs.

Models in which X is a finite-state Markov chain are given a direct computation in which the principal eigenvalue calculation reduces to finding an eigenvector of a matrix with all positive entries.

Example 7.6

The stochastic process X is governed by a finite-state Markov chain on state space $\{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n\}$, where $\mathbf{s}_i$ is the $n \times 1$ vector whose components are all zero except for 1 in the i^{th} row. The transition matrix is $\mathbb{P}$, where $\mathbf{p}_{ij} = \text{Prob}(X_{t+1} = \mathbf{s}_j | X_t = \mathbf{s}_i)$. We represent the Markov chain as

$$X_{t+1} = \mathbb{P}' X_t + W_{t+1}$$

where $\mathbb{E}(X_{t+1} | X_t) = \mathbb{P}' X_t$, $\mathbb{P}'$ denotes the transpose of $\mathbb{P}$, and W_{t+1} is an $n \times 1$ vector process that satisfies $\mathbb{E}(W_{t+1} | X_t) = 0$, which is therefore a martingale-difference sequence adapted to $X_t, X_{t-1}, \dots, X_0$. Think of W_{t+1} as the vector of errors when forecasting X_{t+1} based on current information.

Let $\mathbb{G}$ be an $n \times n$ matrix whose (i, j) entry $\mathbf{g}_{ij}$ is an additive contribution to the growth that $Y_{t+1} - Y_t$ experiences when $X_{t+1} = \mathbf{s}_j$ and $X_t = \mathbf{s}_i$. The stochastic process Y is governed by the *additive functional*

$$Y_{t+1} - Y_t = (X_t)' \mathbb{G} X_{t+1} = (X_t)' \mathbb{G} \mathbb{P}' X_t + (X_t)' \mathbb{G} W_{t+1}$$

Let $M = \exp(Y)$. Define a matrix $\mathbb{M}$ whose $(i, j)^{th}$ element is $\mathbf{m}_{ij} = \exp(\mathbf{g}_{ij})$. The stochastic process M is governed by the *multiplicative functional*:

$$\frac{M_{t+1}}{M_t} = \exp \left[(X_t)' \mathbb{G} X_{t+1} \right] = (X_t)' \mathbb{M} X_{t+1}. \quad (7.8)$$

Associated with this multiplicative functional is the principal eigenvalue problem

$$\mathbb{E} \left[\frac{M_{t+1}}{M_t} \tilde{e} \cdot X_{t+1} | X_t = x \right] = \exp(\tilde{\eta}) \tilde{e} \cdot x.$$

To convert this to a linear algebra problem, write the j^{th} entry of $\tilde{e}$ as $\tilde{e}_j$. Since X_t always assumes the value of one of the coordinate vectors $\mathbf{s}_i, i = 1, \dots, n$,

$$(X_t)' \mathbb{M} X_{t+1} = \mathbf{m}_{ij}$$

when $X_t = \mathbf{s}_i$ and $X_{t+1} = \mathbf{s}_j$. This allows us to rewrite the principal eigenvalue problem as

$$\sum_j \mathbf{p}_{ij} \mathbf{m}_{ij} \tilde{e}_j = \exp(\tilde{\eta}) \tilde{e}_i$$

or

$$\tilde{\mathbb{M}} \tilde{e} = \exp(\tilde{\eta}) \tilde{e} \quad (7.9)$$

where $\tilde{\mathbf{m}}_{ij} \stackrel{\text{def}}{=} \mathbf{p}_{ij} \mathbf{m}_{ij}$ and $\tilde{e}_i$ is entry i of $\tilde{e}$. We want the largest eigenvalue and the one associated with a positive eigenvector of [\(7.9\)](#).

After solving the principal eigenvalue problem, compute

$$\tilde{\mathbf{l}}_{ij} = \exp(-\tilde{\eta}) \tilde{\mathbf{m}}_{ij} \frac{\tilde{e}_j}{\tilde{e}_i} \quad (7.10)$$

and form the matrix $\tilde{\mathbb{L}} = [\tilde{\mathbf{l}}_{ij}]$. We have now constructed a matrix $\tilde{\mathbb{L}}$ that behaves as a transition matrix for a different finite-state Markov chain. Its entries are nonnegative, and

$$\sum_{j=1}^n \tilde{\mathbf{l}}_{ij} = 1.$$

We can use this matrix for form increments $(X_t)' \tilde{\mathbb{L}} X_{t+1}$ in a positive multiplicative martingale process $\{\tilde{L}_t\}$:

$$\tilde{N}_{t+1} = (X_t)' \tilde{\mathbb{L}} X_{t+1}.$$

To achieve a [Theorem 7.1](#) representation of the multiplicative functional M_t , use formula [\(7.10\)](#) for $\tilde{m}_{ij}$ to get $m_{ij} = \exp(\tilde{\eta}) \tilde{m}_{ij} \frac{e_i}{e_j}$. This allows us to write [\(7.8\)](#) as

$$\frac{M_{t+1}}{M_t} = \exp(\tilde{\eta}) \left[(X_t)' \tilde{\mathbb{M}} X_{t+1} \right] \left(\frac{\tilde{e} \cdot X_t}{\tilde{e} \cdot X_{t+1}} \right). \quad (7.11)$$

Remark 7.5

Using the previous example, note that the matrix $\tilde{\mathbb{M}}$ could be inferred from the so-called Arrow prices. Specifically, given prices of all of the state contingent claims for next period, we can infer the column of $\tilde{\mathbb{M}}$ associated with the current state. By observing prices for all of the current states, all of the columns of $\tilde{\mathbb{M}}$. We only need to know $\tilde{\mathbb{M}}$ to pose and solve the principal eigenvalue problem and hence to construct transition probabilities as given by [\(7.10\)](#), but not the $\mathbf{p}_{ij}$'s and the $\mathbf{s}_{ij}$. This gives us a way to *recover* a transition probabilities from asset prices in the language of [\[Ross, 2015\]](#). Notice, in particular, that the $\mathbf{p}_{ij}$'s and $\mathbf{s}_{ij}$'s cannot be inferred uniquely from the $\mathbf{m}_{ij}$'s. In particular, the Arrow prices are insufficient to *identify* the transition probabilities. Ross imposed restrictions on $\mathbf{s}_{ij}$'s that implied that the recovered probabilities are actually the $\mathbf{p}_{ij}$'s. In general, the recovered probabilities will differ from the baseline transition probability matrix. We will have more to say about this difference and investigate why the recovered probabilities remain interesting. See [\[Borovička et al., 2016\]](#) for an extended discussion of this claim.

The following log-linear, log-normal specification displays relevant mechanics of the multiplicative factorization with direct connection to the corresponding additive decomposition

Example 7.7

Consider a stationary X process and an additive Y process described by the VAR

$$\begin{aligned} X_{t+1} &= \mathbb{A}X_t + \mathbb{B}W_{t+1} \\ Y_{t+1} - Y_t &= \nu + \mathbb{D}X_t + \mathbb{F}W_{t+1} \end{aligned}$$

where $\mathbb{A}$ is a stable matrix and $\{W_{t+1} : t \geq 0\}$ is a sequence of independent and identically normally distributed random vectors with mean zero and covariance matrix $\mathbb{I}$. In [Proposition 4.1](#) of [Chapter 4](#), we described the decomposition

$$Y_t - Y_0 = t\nu + \left[\sum_{j=1}^t \mathbb{H}W_j \right] - g(X_{t-1}) + g(X_0) \quad (7.12)$$

where

$$\begin{aligned} \mathbb{H} &= \mathbb{F} + \mathbb{D}(\mathbb{I} - \mathbb{A}')^{-1}\mathbb{B} \\ g(x) &= \mathbb{D}(\mathbb{I} - \mathbb{A})^{-1}x. \end{aligned}$$

Let $M_t = \exp(Y_t)$. Use equation (7.12) to deduce

$$\frac{M_t}{M_0} = \exp(\tilde{\eta}t) \tilde{L}_t \left[\frac{\tilde{e}(X_0)}{\tilde{e}(X_t)} \right]$$

where

$$\tilde{\eta} = \nu + \frac{|\mathbb{H}|^2}{2},$$

$$\tilde{N}_{t+1} = \exp\left(\mathbb{H}W_{t+1} - \frac{|\mathbb{H}|^2}{2}\right), \quad \tilde{L}_0 = 1, \quad (7.13)$$

and

$$\tilde{e}(x) = \exp[g(x)] = \exp\left[\mathbb{D}(\mathbb{I} - \mathbb{A})^{-1}x\right].$$

While the additive martingale of $Y = \log(M)$ has a variance that grows linearly over time, this variance contributes a component to the exponential trend of the multiplicative functional M along with an adjustment to the martingale component.

Example 7.8

Consider the following stochastic volatility example. Suppose that a scalar state variable evolve as first-order autoregression:

$$\begin{aligned} X_{t+1} &= \mathbf{a}X_t + \mathbf{b}W_{t+1} \\ Y_{t+1} - Y_t &= \nu + (\mathbf{f}_0 + \mathbf{f}_1X_t)W_{t+1} \end{aligned}$$

where $\{W_{t+1} : t \geq 0\}$ is an iid sequence of normally distributed random variables. The state variable X_t gives the source of volatility fluctuations. Guess a principle eigenfunction of the form:

$$\tilde{e}(x) = \exp \left(\epsilon_1 x + \frac{\epsilon_2}{2} x^2 \right),$$

and construct

$$\hat{e}(x, w) \stackrel{\text{def}}{=} \epsilon_1 \mathbf{a}x + \epsilon_1 \mathbf{b}w + \frac{\epsilon_2}{2} \mathbf{a}^2 x^2 + \epsilon_2 \mathbf{a}\mathbf{b}xw + \frac{\epsilon_2}{2} \mathbf{b}^2 w^2.$$

Write the principle eigenfunction equation as:

$$\begin{aligned} \log \mathbb{E} \left(\exp \left[\nu + (\mathbf{f}_0 + \mathbf{f}_1 X_t) W_{t+1} + \hat{e}(X_t, W_{t+1}) \right] \mid X_t = x \right) \\ = \tilde{\eta} + \epsilon_1 x + \frac{\epsilon_2}{2} x^2, \end{aligned}$$

where we took logarithms of both sides of the equation. The computation on left side of this equation has a tractable formula for expressing the outcome as a quadratic function of the state.

We may solve this equation in three steps. Solve the equation in three steps. First, equate coefficients on x^2 and deduce a quadratic equation for ϵ_2 . Next, equate coefficients on x and obtain a linear equation for ϵ_1 . Finally, equate constants obtain an equation for $\tilde{\eta}$. The initial quadratic equation is

$$\mathbf{b}^2(\epsilon_2)^2 + (\mathbf{a}^2 + 2\mathbf{f}_1\mathbf{a}\mathbf{b} - 1)\epsilon_2 + (\mathbf{f}_1)^2 = 0.$$

When it has a solution, there is will typically be two possible choices. For the solution to be of interest, $1 - \epsilon_2 \mathbf{b}^2$ must be positive.

For this example,

$$\tilde{N}_{t+1} \propto \exp \left[(\mathbf{f}_0 + \mathbf{f}_1) X_t W_{t+1} + \epsilon_1 \mathbf{b} W_{t+1} + \frac{\epsilon_2}{2} \mathbf{b}^2 (W_{t+1})^2 + \epsilon_2 \mathbf{a}\mathbf{b} X_t W_{t+1} \right]$$

where $\propto$ means proportional to up to a scale factor that depends on X_t . Under the implied change in measure, W_{t+1} is distributed as normal random variable with conditional mean

$$\frac{\mathbf{f}_0 + \mathbf{f}_1 X_t + \epsilon_1 \mathbf{b} + \epsilon_2 \mathbf{a}\mathbf{b} X_t}{1 - \epsilon_2 \mathbf{b}^2}$$

and precision $1 - \epsilon_2 \mathbf{b}^2$. Under the change of probability measure, W_{t+1} remains normally distributed with a different variance and state-dependent mean. Under this change in probability measure, the X remains a first-order autoregression but with different dynamics. It could, for instance, be an explosive autoregression.

Remark 7.6

The martingale component of the multiplicative functional has “peculiar behavior.” It has expectation one by construction. The Martingale Convergence Theorem guarantees that sample paths converge, typically to zero. The Martingale Convergence Theorem is operative because the positive martingale is bounded from below. Since its date zero conditional expectation is one, for long horizons this process necessarily has a fat right tail. [Fig. 7.2](#) plots probability density functions of the martingale component for different values of t .

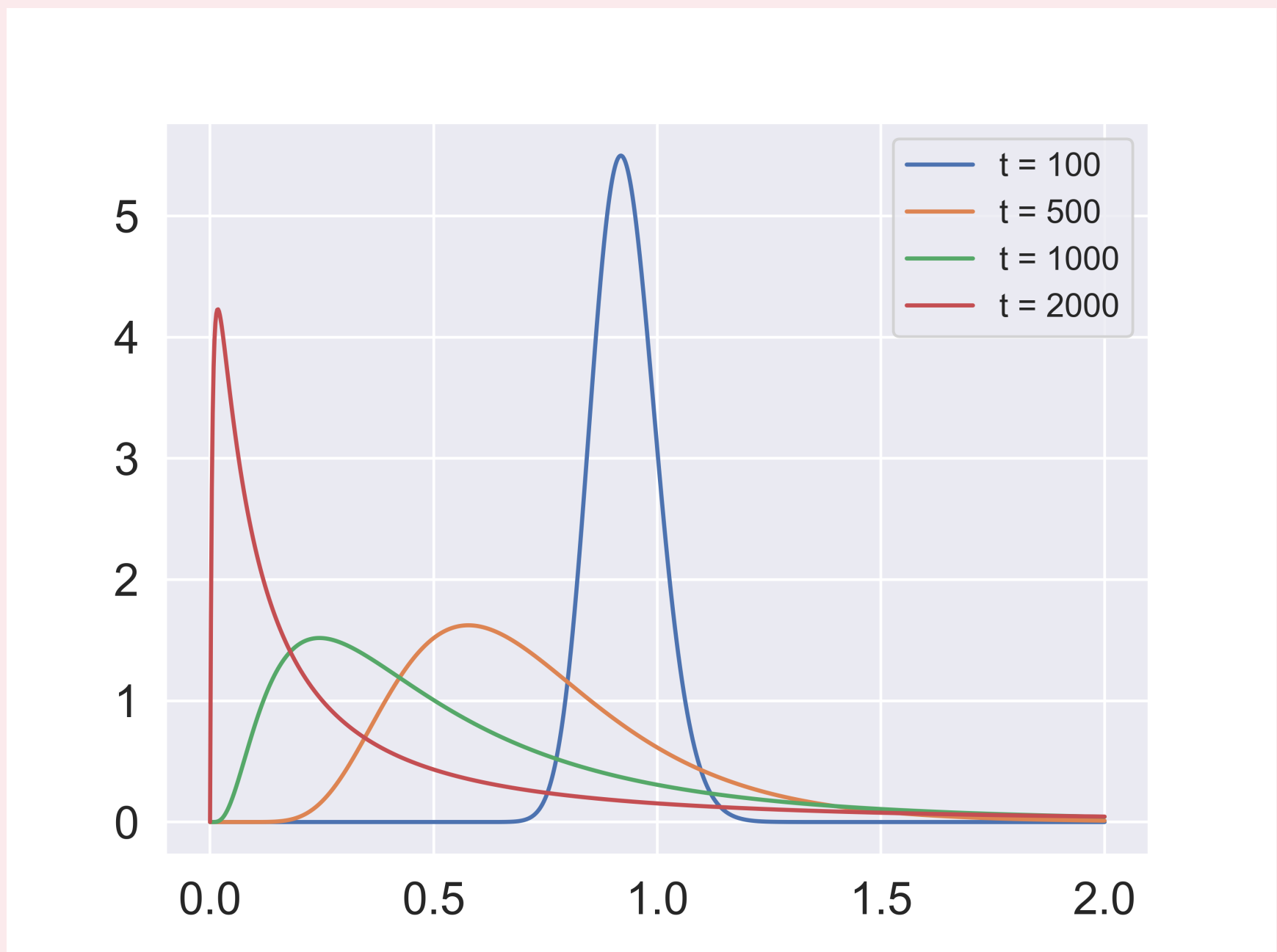


Fig. 7.2 Density of $\tilde{L}_t$ for different values of t .

Remark 7.7

Consider a positive cumulative return process, R , modeled as a stochastic functional. As we noted previously, for such a process, R_t/R_τ for $\tau < t$ is a $t - \tau$ -period return for any such t and τ . Consider also the associated cumulative stochastic discount factor process, S , which we also take to be a multiplicative functional. For this process, S_t/S_τ for $\tau < t$ is the $t - \tau$ -period stochastic discount factor for any such t and τ . Normalize $R_0 = 1$ and $S_0 = 1$. Repeating what we stated previously, by standard asset pricing logic, SR is a multiplicative martingale. [\[Martin, 2012\]](#) studies tail behavior of cumulative returns. Since its date zero conditional expectation is one, for long horizons this process necessarily has a fat right tail as illustrated by [Fig. 7.2](#).

While the multiplicative martingale has peculiar sample path properties, we are primarily interested in this martingale component as a change of probability measure. For instance, in [Example 7.7](#), formula [\(7.13\)](#) for $\tilde{N}_t$ tells how the change in probability measure induces mean $\mathbb{H}$ in the conditional distribution for the shock W_{t+1} . Similarly, $\tilde{\mathbb{L}}$, with entries given in formula [\(7.10\)](#), provides an alternative transition matrix in [Example 7.6](#).

7.5. Stochastic stability

Our characterization of a change of probability measure as the solution of a Perron-Frobenius problem determines only transition probabilities. Since the process is Markov, it is of interest to seek an initial distribution of X_0 under which the process is stationary and satisfies a stochastic stability property that we will define and explore. Stochastic stability opens the door the study of a variety of limiting behavior and hence justifies our interest in the multiplicative factorization. The [eigenfunction problem](#) can have multiple solutions, it turns out, however, that there is a unique solution for which the process X is **stochastically stable** under the implied change of measure, in particular, the solution associated with the minimum eigenvalue. See [[Hansen and Scheinkman, 2009](#)] and [[Hansen, 2012](#)] for formal analyses of this problem in a continuous-time Markov setting.

In what follows we investigate the discrete-time counterpart to their investigations.

Definition 7.3

A process X is **stochastically stable** under a probability measure $\tilde{P}$ if it is stationary and $\lim_{j \rightarrow \infty} \tilde{\mathbb{E}}[h(X_j) \mid X_0 = x] = \tilde{\mathbb{E}}[h(X_0)]$ for any Borel measurable h satisfying $\tilde{\mathbb{E}}|h(X_t)| < \infty$.

Theorem 7.2

Let M be a multiplicative functional. Suppose that $(\tilde{\eta}, \tilde{e})$ solves the [eigenfunction problem](#) and that under the change of measure $\tilde{P}$ implied by the associated martingale $\tilde{M}$ the stochastic process X is stationary and ergodic. Consider any other solution (η^*, e^*) to [eigenfunction problem](#) with implied martingale $\{M_t^*\}$. Then

1. $\eta^* \geq \tilde{\eta}$.
2. If X is stochastically stable under the change of measure Pr^* implied by the martingale M^* , then $\eta^* = \tilde{\eta}$, e^* is proportional to $\tilde{e}$, and $M^* = \tilde{M}$ for all $t = 0, 1, \dots$

Proof

First we show that $\eta^* \geq \tilde{\eta}$. Write:

$$\mathbb{M}^t e^*(x) = \exp(\tilde{\eta}t) \tilde{\mathbb{E}} \left(\left[\frac{\tilde{e}(X_0)}{\tilde{e}(X_t)} \right] e^*(X_t) \middle| X_0 = x \right) = \exp(\eta^*t) e^*(x).$$

Thus,

$$\tilde{\mathbb{E}} \left(\left[\frac{e^*(X_t)}{\tilde{e}(X_t)} \right] \middle| X_0 = x \right) = \exp(\eta^*t - \tilde{\eta}t) \left[\frac{e^*(x)}{\tilde{e}(x)} \right].$$

If $\tilde{\eta} > \eta^*$, then

$$\lim_{t \rightarrow \infty} \tilde{E} \left(\left[\frac{e^*(X_t)}{\tilde{e}(X_t)} \right] \middle| X_0 = x \right) = 0.$$

But this equality cannot be true because under $\widetilde{Pr}$ X is stochastically stable and $\frac{e^*}{\tilde{e}}$ is strictly positive. Therefore, $\eta^* \geq \tilde{\eta}$.

Consider next the case in which $\eta^* > \tilde{\eta}$. Write

$$\frac{M_t}{M_0} = \exp(\eta^*t) \left(\frac{M_t^*}{M_0^*} \right) \left(\frac{e^*(X_0)}{e^*(X_t)} \right),$$

which implies that

$$\mathbb{M}^t \tilde{e}(x) = \exp(\eta^*t) E^* \left[\left(\frac{e^*(X_0)}{e^*(X_t)} \right) \tilde{e}(X_t) \middle| X_0 = x \right] = \exp(\tilde{\eta}t) \tilde{e}(x).$$

Thus,

$$\mathbb{E}^* \left(\left[\frac{\tilde{e}(X_t)}{e^*(X_t)} \right] \middle| X_0 = x \right) = \exp(\tilde{\eta}t - \eta^*t) \left[\frac{\tilde{e}(x)}{e^*(x)} \right].$$

Suppose that $\tilde{\eta} < \eta^*$, then

$$\lim_{t \rightarrow \infty} \mathbb{E}^* \left(\left[\frac{\tilde{e}(X_t)}{e^*(X_t)} \right] \middle| X_0 = x \right) = 0,$$

so that X cannot be stochastically stable under the Pr^* measure.

Finally, suppose that $\tilde{\eta} = \eta^*$ and that $\frac{\tilde{e}(x)}{e^*(x)}$ is not constant. Then

$$\mathbb{E}^* \left(\left[\frac{\tilde{e}(X_t)}{e^*(X_t)} \right] \middle| X_0 = x \right) = \frac{\tilde{e}(x)}{e^*(x)}$$

and X cannot be stochastically stable under the Pr^* measure.

Stochastic stability under the change of measure provides way to think about some interesting long-term approximations. Suppose that

$$f > 0 \text{ and } 0 < \tilde{\mathbb{E}} \left[\frac{f(X_t)}{\tilde{e}(X_t)} \right] < \infty. \quad (7.14)$$

Then

$$\frac{1}{j} \log \mathbb{M}^j f(x) = \tilde{\eta} + \frac{1}{j} \log \tilde{\mathbb{E}} \left[\frac{f(X_j)}{\tilde{e}(X_j)} \middle| X_0 = x \right] - \frac{1}{j} \log \tilde{e}(x)$$

Since X is stochastically stable under $\tilde{Pr}$,

$$\lim_{j \rightarrow \infty} \frac{1}{j} \log \mathbb{M}^j f(x) = \tilde{\eta}. \quad (7.15)$$

This limit justifies the Perron Frobenius eigenvalue as the long-term growth or decay rate of the multiplicative functional.

As we have mentioned previously, cumulative returns and cumulative stochastic discount factor processes can often be measured, conveniently, by multiplicative functionals. Suppose that

$$\tilde{\mathbb{E}} \left(\frac{1}{\tilde{e}} \right) < \infty.$$

In the case of a cumulative return process, the implied $\tilde{\rho} > 0$ is the long-term logarithm of the expected return on the corresponding investment. The division by j in (7.15) adjusts for the horizon of the expected return. For a cumulative stochastic discount factor process, the resulting $\tilde{\rho} < 0$ is the negative of the yield on a long-term discount bond.

Under the restriction, after adjusting for the growth or decay in the semigroup, we obtain a more refined approximation:

$$\lim_{j \rightarrow \infty} \exp(-\tilde{\eta}j) \mathbb{M}^j f(x) = \lim_{j \rightarrow \infty} \tilde{\mathbb{E}} \left[\frac{f(X_j)}{\tilde{e}(X_j)} \middle| X_0 = x \right] \tilde{e}(x) = \tilde{E} \left[\frac{f(X_t)}{\tilde{e}(X_t)} \right] \tilde{e}(x),$$

where we assume that $\tilde{\mathbb{E}} \left[\frac{f(X_t)}{\tilde{e}(X_t)} \right] < \infty$. Once we adjust for the impact of $\tilde{\eta}$, the limiting function is proportional to $\tilde{e}$. The function f determines only a scale factor $\tilde{\mathbb{E}} \left[\frac{f(X_t)}{\tilde{e}(X_t)} \right] \tilde{e}(x)$.

We will apply the limiting results in a variety of ways in this and subsequent chapters. The multiplicative factorization can help us understand implications of stochastic equilibrium models for valuations of random payout processes. In addition, such factorizations can help organize empirical evidence in ways that make contact with such stochastic equilibrium asset pricing models.

7.6. Inferences about permanent shocks

Macroeconomists often study dynamic impacts of shocks to systems of variables measured in logarithms. For example, [\[Alvarez and Jermann, 2005\]](#) suggest looking at asset prices using a multiplicative representation of a cumulative stochastic discount factor, though without the tools provided by this chapter. The additive decomposition derived and analyzed in [Chapter 4](#) are a convenient tool for models like theirs.

We start with a factorization of a stochastic discount factor process as given in [Theorem 7.1](#).

$$\frac{S_t}{S_0} = \exp(t\eta^s) L_t^s \left[\frac{e^s(X_0)}{e^s(X_t)} \right] \quad (7.16)$$

Take logarithms and form:

$$\log S_t - \log S_0 = t\eta^s + \log L_t^s + \log e^s(X_0) - \log e^s(X_t).$$

This looks like an additive decomposition of the type analyzed in [Chapter 4](#), but it is actually different. While L^s is a multiplicative martingale, $\log L^s$ is typically a super martingale, but not a martingale. This leads us to write the additive decomposition as:

$$\log S_t - \log S_0 = t\hat{\eta}^s + \hat{L}_t^s + \hat{e}^s(X_0) - \hat{e}^s(X_t)$$

where $\hat{L}^s$ is an additive martingale. As [\[Hansen, 2012\]](#) argues, a weaker result holds. If L^s is not degenerate (i.e., equal to one), then $\hat{L}^s$ is not degenerate and conversely. A prominent multiplicative martingale component implies a prominent role for permanent shocks in the underlying economic dynamics. A formal probability model lets us link the two representations via the results we have described in this chapter and [Chapter 4](#). Log normal models, at least as approximations, are often used by applied macroeconomists. [Example 7.7](#) provides an example with explicit formulas linking the two representations. While distinct, the two martingales, in this special case, are closely linked. In general, this simplicity vanishes, unfortunately.

7.7. Observable counterparts to a factorization of stochastic discount factors

Consider stochastic discount factorization [\(7.16\)](#) again. A date zero price of a long-term bond is:

$$\mathbb{E} \left(\frac{S_t}{S_0} \middle| \mathfrak{A}_0 \right) = \exp(\eta^s t) \tilde{\mathbb{E}} \left[\frac{1}{e^s(X_t)} \middle| X_0 \right] e^s(X_0).$$

Compute the corresponding yield by taking $1/t$ times minus the logarithm:

$$-\eta^s - \frac{1}{t} \log \tilde{\mathbb{E}} \left[\frac{1}{e^s(X_t)} \middle| X_0 \right] + \frac{1}{t} \log e^s(X_0).$$

Provided that

$$\tilde{\mathbb{E}} \left[\frac{1}{e^s(X_t)} \right] < \infty,$$

the limiting yield on a discount bond is $-\eta^s$, as we noted previously.

Next consider a one-period holding period return on a t period discount bond:

$$\frac{\exp[\eta^s(t-1)] \tilde{\mathbb{E}} \left[\frac{1}{e^s(X_t)} \middle| X_1 \right] e^s(X_1)}{\exp(\eta^s t) \tilde{\mathbb{E}} \left[\frac{1}{e^s(X_t)} \middle| X_0 \right] e^s(X_0)}$$

Using stochastic stability and taking limits as t tends to ∞ gives the limiting holding-period return:

$$R_1^\infty \stackrel{\text{def}}{=} \exp(-\eta^s) \frac{e^s(X_1)}{e^s(X_0)}. \quad (7.17)$$

A simple calculation shows that R_1^∞ satisfies the following equilibrium pricing restriction on a one-period return:

$$\mathbb{E} \left[\left(\frac{S_1}{S_0} \right) R_1^\infty \middle| \mathfrak{A}_0 \right] = \tilde{\mathbb{E}} \left(\exp(\eta^s) \left[\frac{e^s(X_0)}{e^s(X_1)} \right] \exp(-\eta^s) \left[\frac{e^s(X_1)}{e^s(X_0)} \right] \middle| X_0 \right) = 1.$$

These long-horizon limits provide approximations to the eigenvalue for the stochastic discount factor and the ratio of the eigenfunctions. In a model without a martingale component [[Kazemi, 1992](#)], observed that the inverse of this holding-period return is the one-period stochastic discount factor. [[Alvarez and](#)

[Jermann, 2005](#)] extend this insight by showing that the reciprocal reveals the component of one-period stochastic discount factor net of its martingale component.

 Remark 7.8

Within the [\[Kazemi, 1992\]](#) setup, R_1^∞ equals the one-period stochastic factor. Let Y_1 be a vector of asset payoffs and Q_0 is a vector of corresponding prices. Then

$$\mathbb{E}(R_1^\infty Y_1 \mid \mathcal{A}_0) = Q_0.$$

Take the baseline probability measure to the data generating process as is often done under rational expectations. We may sidestep measuring this baseline probability by using the approach delineated by [\[Hansen and Singleton, 1982\]](#) and [\[Hansen and Richard, 1987\]](#) and converts our generalization of Kazemi's insight into a testable restriction using generalized method of moments. See [\[Hansen, 1982\]](#).

Notice that rational expectations is imposed empirically under *partial information*, as it often is for Euler equation methods. This is very distinct from imposing rational expectations as an outcome determined by a fully-specified model. This *full information* approach can be imposed without feeding back on data. The expectations are *model implied*. Indeed many applied theory contributions embrace this approach to rational expectations. With the full information approach, maximum likelihood estimation and testing can be done as a second step in the analysis, as is featured in research by Sargent and others. See, for instance, the methods described in [\[Hansen and Sargent, 1980\]](#).

Within the [\[Kazemi, 1992\]](#) setup, a subjective belief specification as multiplicative martingale expressed in terms of a baseline probability specification could rationalize the martingale component of a cumulative stochastic discount factor process and capture the empirical failure using the approach just described. Thus the distinction between probability measures can be sufficient to induce a martingale component relative to baseline probabilities even though this component is absent when valuations are depicted with the subjective probabilities.

In practice, we have only have bond data with a finite payoff horizon, whereas the characterizations [\[Kazemi, 1992\]](#) and [\[Alvarez and Jermann, 2005\]](#) use bond prices with a limiting payoff horizon. Empirical implementations using such characterizations assume that the observed term structure data have a sufficiently long duration component to provide plausible proxy for the limiting counterpart.

7.8. Long-term risk-return tradeoff for cash flows

Following [\[Hansen and Scheinkman, 2009\]](#) and [\[Hansen et al., 2008\]](#), we consider the valuation of stochastic cash flows, G , that are multiplicative functionals. Such cash flows are determinants of prices of both equities and bonds.

We now study long-term limits of prices of such cash flows. In addition to the stochastic discount process [\(7.16\)](#), form:

$$\log G_t - \log G_0 = t\eta^g + \log L_t^g + \log e^s(X_0) - \log e^s(X_t),$$

with a corresponding cash-flow return over horizon t :

$$\frac{G_t}{\mathbb{E} \left[\left(\frac{S_t}{S_0} \right) G_t \middle| \mathfrak{A}_0 \right]} = \frac{\frac{G_t}{G_0}}{\mathbb{E} \left(\frac{S_t G_t}{S_0 G_0} \middle| X_0 \right)}.$$

Note that as a special case, the cash-flow return on a unit date t cash-flow is:

$$\frac{1}{\mathbb{E} \left(\frac{S_t}{S_0} \middle| X_0 \right)}$$

Define the *proportional risk premium* on the initial cash-flow return as the ratio of the expected return divided by the riskless counterpart for the same horizon. Taking logarithms and adjusting for the time horizon gives:

$$\frac{1}{t} \log \mathbb{E} \left(\frac{G_t}{G_0} \middle| X_0 \right) - \frac{1}{t} \log \mathbb{E} \left(\frac{S_t G_t}{S_0 G_0} \middle| X_0 \right) + \frac{1}{t} \log \mathbb{E} \left(\frac{S_t}{S_0} \middle| X_0 \right), \quad (7.18)$$

where the first term is the logarithm of the expected payoff, the second term is minus the logarithm of the price, and the third term is minus the logarithm of the riskless cash-flow return for horizon t . Scaling by $1/t$ adjusts for the investment horizon.

The product SG is itself a multiplicative functional. Let η^{sg} denote its geometric growth component. Then from [\(7.18\)](#), the limiting cash-flow risk compensation is:

$$\eta^g + \eta^s - \eta^{gs}.$$

This expression resembles the negative of covariance as is often found in asset pricing, but it differs from a covariance because we are working with proportional measures of the risk compensations.

We also investigate the limiting behavior of one-period holding period returns. An empirical asset pricing literature has explored these returns starting with [[van Binsbergen et al., 2012](#)]. See [[Golez and Jackwerth, 2024](#)] for a recent update of this evidence. Use the factorization of SG to get

$$\begin{aligned} \log G_t - \log G_0 + \log S_t - \log S_0 \\ = \eta^{sg} + \log L_t^{sg} + \log e^{gs}(X_0) - \log e^{gs}(X_t), \end{aligned}$$

Based as it is on a multiplicative factorization of SG , this typically does not the difference between the logarithm of the factorization of S and the logarithm of the factorization of G . We provide a characterization of the limiting one-period holding period return for the cash flow by imitating and extending our analysis of a limiting holding-period return for riskless bond. This gives the following cash-flow analogue to [\(7.17\)](#):

$$\exp(-\eta^{sg}) \left(\frac{G_1}{G_0} \right) \left[\frac{e^{sg}(X_1)}{e^{sg}(X_0)} \right]$$

for the long-horizon cash flow holding period return. The eigenvalue and eigenfunction adjustments come from studying SG instead of S . The limiting holding-period return now inherits a stochastic growth term G_1/G_0 . By multiplying this return by S_1/S_0 we obtain N_1^{sg} , the date one martingale increment for SG . The one-period pricing relation for the cash-flow holding-period return follows immediately.

Finally, suppose that $L^s = 1$ so that the martingale component of the stochastic discount factor process is degenerate. Then SG inherits the martingale component of G , implying that

$$\begin{aligned} \eta^{sg} &= \eta^s + \eta^g \\ e^{sg}(x) &= e^s(x)e^g(x) \end{aligned}$$

As a consequence, the long-term risk-return tradeoff is zero, since in the limit proportional risk compensation is

$$\eta^s + \eta^g - \eta^{sg} = 0.$$

Thus when the stochastic discount factor process fails to have a martingale component, the long-term, risk-return tradeoff is *degenerate*.

7.9. Bounding investor beliefs

We use the cumulative stochastic discount factorization to analyze two distance approaches to drawing inferences about investor beliefs.

7.9.1. Subjective beliefs in the absence of long-term risk

Suppose that we have data on prices of one-period state-contingent claims. We can use these data to infer the one-period operator, $\mathbb{M}$. Recall that we represent this operator using a baseline specification of the one-period transition probabilities. One possibility is that the one-period baseline transition probabilities agree with the data generation. Rational expectations models equate transition probabilities to those used by investors. More generally, investors could have subjective beliefs that can differ from the baseline specification:

- investors think there are no permanent macroeconomic shocks;
- investors don't have risk-based preferences that can induce a multiplicative martingale in a cumulative stochastic discount factor process.^[1]

Under these two restrictions, we could identify the L^s as the likelihood ratio for investor beliefs relative to the baseline probability distribution. Thus, the implied martingale component in the cumulative stochastic discount factor identifies the subjective beliefs of investors. Using this change of measure, the limiting long-term risk compensations derived in the previous section are zero. These assumptions allow for the "Ross recovery" of investor beliefs.^[2]

7.9.2. Restricting the martingale increment with limited asset market data

Suppose we impose rational expectations by endowing investors with knowledge of the data generating process. With limited asset market data we cannot identify the martingale component to cumulative stochastic discount factor process without additional model restrictions. We can, however, obtain potentially useful bounds on the martingale increment. For some applications, in addition to that of [\[Alvarez and Jermann, 2005\]](#), see [\[1\]](#) and [\[2\]](#), among others. We know that as a stochastic process the implied martingale has some peculiar behavior and the martingale and transient components can be correlated the transient components. Nevertheless, the implied probability measure can be well behaved. Consequently, in contrast to [\[Alvarez and Jermann, 2005\]](#), [\[1\]](#), and [\[2\]](#), we use the increment as a device to represent conditional probabilities instead of just as a random variable. Extensions of these same methods can be used to study restrictions on subjective beliefs implied by asset prices.

There is a substantial literature on divergence measures for probability densities. Relative entropy is an important example. More generally, consider a convex function ϕ that is zero when evaluated at one. Jensen's inequality implies that

$$\mathbb{E} [\phi(N_1) \mid X_0] \geq 0,$$

and equal to zero when N_1 is one, provided that N_1 is a multiplicative martingale increment (has conditional expectation one). This gives rise to a family of ϕ divergences that can be used to assess

departures from baseline probabilities. Relative entropy, $\phi(n) = n \log n$ is an example that is particularly tractable and has been used often. Both $n \log n$ and $-\log n$ can be interpreted as expected log-likelihood ratios.

One way of assessing the magnitude of N_1^L solves:

Minimum divergence Problem

$$\min_{N_1 \geq 0} \mathbb{E} [\phi(N_1) \mid X_0]$$

subject to:

$$\begin{aligned} \mathbb{E}(N_1 \mid X_0) &= 1 \\ \mathbb{E} \left(N_1 \left[\exp(\eta^s) \frac{e^s(X_0)}{e^s(X_1)} \right] Y_1 \mid X_0 \right) &= Q_0 \end{aligned}$$

where Y_1 is a vector of asset payoffs and Q_0 is a vector of corresponding prices.

Recall that the term

$$\left[\exp(\eta^s) \frac{e^s(X_0)}{e^s(X_1)} \right] = (R_1^\infty)^{-1}$$

can be approximated by the reciprocal of the one-period holding-period return on a long-term bond. This makes such an approach implementable. Such a computation is an example of **partial identification** because the vector Y_1 of asset payoffs may not be sufficient to reconstruct all potential one-period asset payoffs and prices. This could be because data limitations lead an econometrician to choose to use incomplete data on financial markets.

By relaxing rational expectations and replacing $\left[\exp(\eta^s) \frac{e^s(X_0)}{e^s(X_1)} \right]$ in this minimization problem with a candidate one-period stochastic discount factor, S_1/S_0 , the bounds apply to the subjective beliefs of the investors.

Remark 7.9

To avoid having to estimate conditional expectations, applications often study an unconditional counterpart to this problem. In such situations, conditioning can be brought in through the “back door” by scaling payoffs and prices with variables in the conditioning information set; for example, see [[Hansen and Singleton, 1982](#)] and [[Hansen and Richard, 1987](#)]. See [[Bakshi and Chabi-Yo, 2012](#)] and [[Bakshi et al., 2017](#)] for some related implementations.

 Remark 7.10

[[Alvarez and Jermann, 2005](#)] use $-\mathbb{E}(\log S_1 + \log S_0)$ as the objective to be minimized. Notice that

$$\log N_1^s = \log S_1 - \log S_0 + [\eta^s + \log e^s(X_1) - \log e^s(X_0)],$$

where the term in square brackets is the logarithm of the limiting holding-period bond return. The criterion thus equals that in the minimum divergence problem, but with an additive translation. Rewrite the constraints as:

$$\begin{aligned}\mathbb{E}\left(\frac{S_1}{S_0}R_1^\infty\right) &= 1 \\ \mathbb{E}\left(\frac{S_1}{S_0}Y_1\right) &= Q_0.\end{aligned}$$

Thus we are left with an equivalent minimization problem in which the translation term is subtracted off to obtain the bound of interest.

Applied researchers have sometimes omitted the first constraint, which weakens the bound. Moreover, [[Chen et al., 2024](#)] isolate a potentially problematic aspect of monotone decreasing divergences such as $-\log n$ because they can fail to detect certain limiting forms of deviations from baseline probabilities.

 Remark 7.11

[[Chen et al., 2020](#)] propose extensions of the one-period divergence measures to multi-period counterparts that remain tractable and enlightening. Their method for accommodating conditioning information for bounding such divergences has a direct extension to the problem considered here.

A figure from [[Chen et al., 2020](#)]:

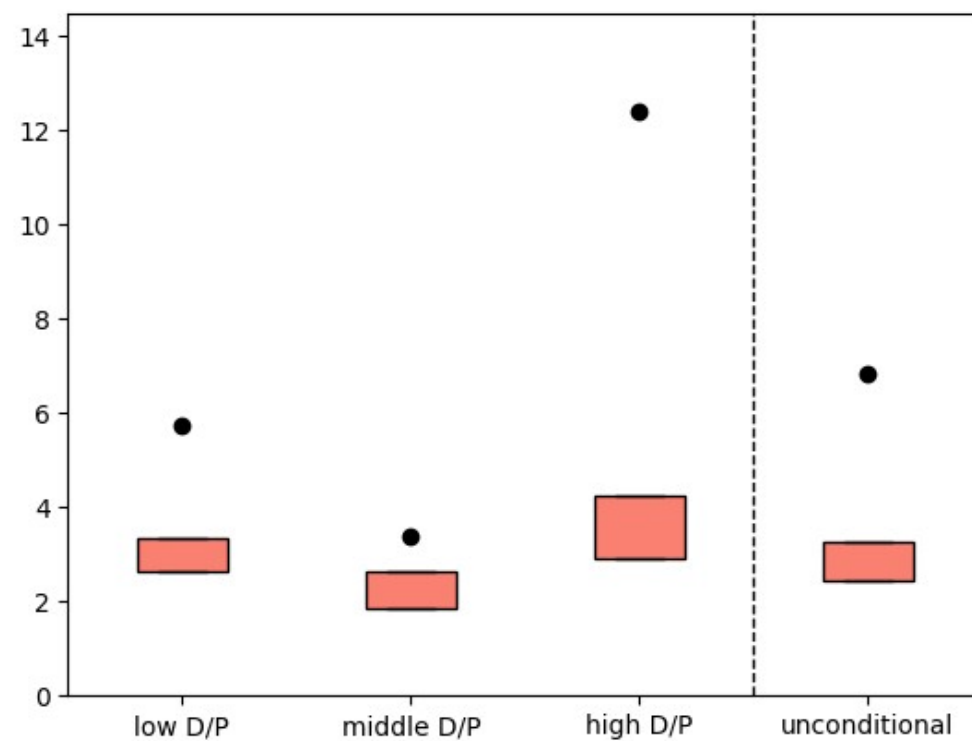


Fig. 7.3 Proportional risk compensations computed as $\log \mathbb{E}R^w - \log \mathbb{E}R^f$ scaled to an annualized percentage. The ●s are the empirical averages and the boxes give the imputed bounds when we inflated the minimum relative entropy by 20%.

-
- [1] This second restriction is violated by recursive utility models of investor preference that we will analyze in a subsequent chapter. See discussions in [\[Alvarez and Jermann, 2005\]](#), [\[Hansen and Scheinkman, 2009\]](#), and [\[Borovička et al., 2016\]](#).
 - [2] Our presentation is consistent with the formal analysis in [\[Ross, 2015\]](#), although Ross's derives and motivates his result somewhat differently.