Likelihood Ratio and Score Processes

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7.1. Introduction

In this chapter we study the behavior of likelihood ratio processes and score processes. We treat these as processes so that we can study their dynamic behavior when additional observations are included. We start by showing how what we call multiplicative martingales imply alternative probability models. We then investigate some limiting behavior that reveals which among multiple models generates the data without imposing an *ex ante* distribution (a prior) over the alternative models. We then provide examples of such martingales constructed from likelihood ratio processes for alternative models. We end by studying the behavior of score processes that are constructed by differentiating log-likelihood with respect to an underlying parameter vector.

7.2. Multiplicative martingales

Let $\{M_t : t \geq 0\}$ denote a process whose logarithm evolves as:

$$\log M_{t+1} - \log M_t = \kappa(X_t, W_{t+1}).$$

Thus the logarithmic counterpart is recognizable as an additive functional as studied in Chapter 4. We call such an $\{M_t : t \geq 0\}$ a multiplicative functional. We explore properties of such processes in some generality in Chapter 8. In this chapter, the processes of interest are multiplicative martingales:

Definition 7.1

The process M is a **multiplicative martingale** if

$$E\left(\exp\left[\kappa(X_t, W_{t+1})\right] \mid X_t\right) = 1,$$

and thus $E\left(M_{t+1} \mid M_t, X_t\right) = M_t$.

Multiplicative martingales provide a convenient way to construct alternative probabilities.

Proposition 7.1

Suppose that $\{M_t : t \geq 0\}$ is a multiplicative martingale. The ratio M_{t+1}/M_t implies a transition probability conditioned on X_t via the formula:

$$\widetilde{E}\left[f(X_t, W_{t+1}) \mid X_t\right] = E\left[\left(\frac{M_{t+1}}{M_t}\right) f(X_t, W_{t+1}) \mid X_t\right].$$

The resulting process for X remains Markovian, and the implied τ-period transition probability can be represented as:

$$\widetilde{E}\left[f(X_{t+\tau-1}, W_{t+\tau}) \mid X_t\right] = E\left[\left(\frac{M_{t+\tau}}{M_t}\right) f(X_{t+\tau-1}, W_{t+\tau}) \mid X_t\right].$$

Proof. The linear operator:

$$\widetilde{E}\left(\cdot \mid X_t = x\right)$$

maps nonnegative functions into nonnegative functions and the unit function into itself. Thus this operator is a conditional expectation. Moreover, it maps functions of (w, x) into functions of x alone. This implies that $\{X_t : t \geq 0\}$ is a first-order Markov process under the implied change of probability. The representation of the implied τ-period conditional expectation operator follows from the Law of Iterated Expectations.

Remark 7.1

Under the change of measure induced by M_{t+1}/M_t, the shock W_{t+1} typically does not have conditional mean zero.

So far we have not restricted the initial condition M_0. We have only characterized its stochastic evolution. Suppose that the process $\{X_t\}$ is stationary with probability denoted Q, and consider $M_0 = \tilde{q}(X_0)$ where $\tilde{q}$ satisfies:

$$
\begin{aligned}
&E\left[M_0 f(X_0)\right] = E\left[M_1 f(X_1)\right] \\
&\int \tilde{q}(x) dQ(x) = 1
\end{aligned}
$$

for all bounded (measurable) functions f of the Markov state x. Then $\widetilde{Q}(dx) \stackrel{\text{def}}{=} \tilde{q}(x)Q(dx)$ is a stationary distribution under the probability measure that M implies.

In what follows we will impose such an initial condition on M_0 in order that we can apply a Law of Large Numbers as characterized in Chapter 1. In addition we will impose the ergodic restriction that the only solutions to the equation:

$$
E\left[\left(\frac{M_{t+1}}{M_t}\right) f(X_{t+1}) \mid X_t = x\right] = f(x)
$$

are constant functions with $\widetilde{Q}$ measure one as investigated in Chapter 2.

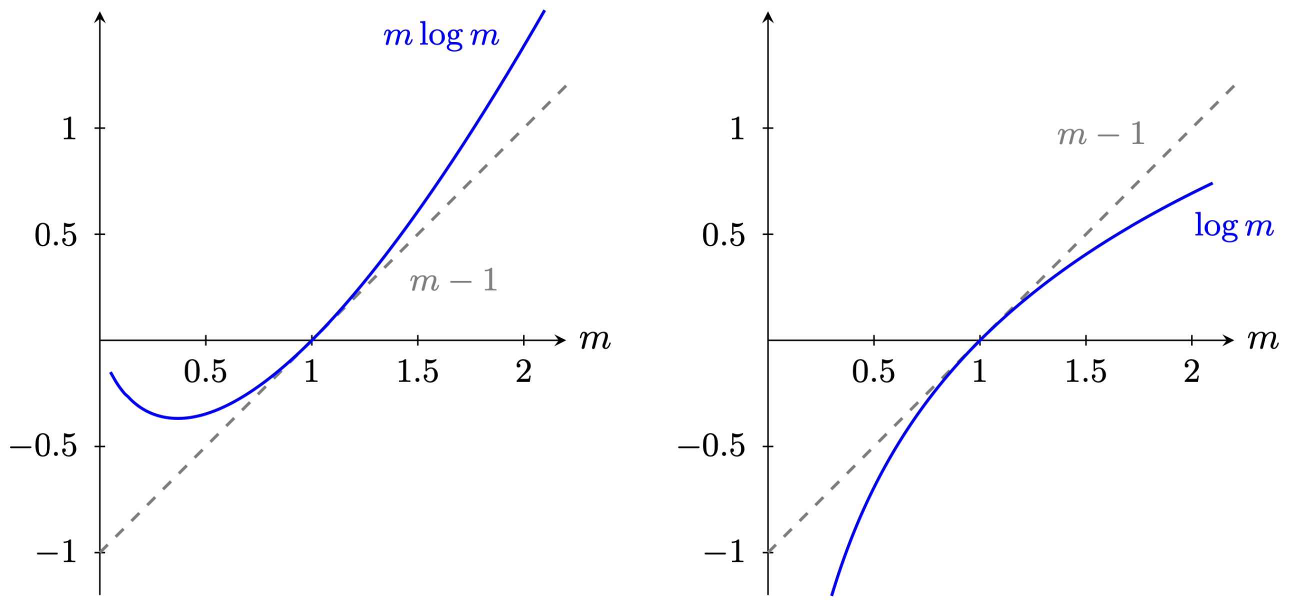
7.3. Multiple models

So far, we have considered two models, an initial one and a second one implied by a multiplicative martingale. Now suppose we have $\ell + 1$ such models, comprising an initial one and ℓ multiplicative martingales. We denote each such martingale as $\{M_t^i : t \geq 0\}$ where each one induces a process that is stationary and ergodic. We may view the initial probability specification as one in which $M_t^0 = 1$ for all $t \geq 0$. This is the ergodic decomposition of Chapter 1. Partition the sample space into $\ell + 1$ invariant events, one matched to each martingale. Each martingale supplies the probabilities conditioned on its own event, and those conditional probabilities are what Definition 1.6 calls a statistical model. Distinguishing among the $\ell + 1$ models is therefore the problem of learning which invariant event occurred.

In this construction, we choose to normalize model zero to be the probability distribution used for representing all of the probabilities in conjunction with multiplicative martingales. Suppose instead we had used model one for this purpose. The $\{M_t^i : t \geq 0\}$ processes cease to be martingales because we have changed the underlying probability. Instead, the ratio processes $\{M_t^i / M_t^1 : t \geq 0\}$ are multiplicative martingales with this change in the underlying probabilities. This outcome of altering the baseline probability will be central in the discussion that follows.

7.4. Some large sample properties

We next consider two gradient inequalities that we will use prominently. The function $m \log m$ is convex and the function $\log m$ is concave. A convex function lies above its gradient approximation and a concave function below its gradient approximation. Observe that the gradient is one for both functions when $m = 1$ and the gradient approximation is $m - 1$. Fig. 7.1 illustrates these two inequalities.



(a) The inequality $m \log m \geq m - 1$. (b) The inequality $m - 1 \geq \log m$.

Fig. 7.1 Gradient inequalities

We use these inequalities in conjunction with conditional expectations to obtain the targets of interest. This amounts to application of what is called Jensen's Inequality and implies

$$
\begin{aligned}
E\left(\log M_{t+1} - \log M_t\right) &\leq 0 \\
E\left[\left(\log M_{t+1} - \log M_t\right)\frac{M_{t+1}}{M_t}\right] &\geq 0.
\end{aligned}
\tag{7.1}
$$

These weak inequalities become strict when $\log M_{t+1} - \log M_t$ is not equal to zero with probability one. Observe that $\{\log M_t : t \geq 0\}$ is an additive functional of the type studied in

Chapter 4 with a trend coefficient that is negative. A consequence of the Law of Large Numbers and the first inequality in (7.1) is that:

$$\lim_{t \to \infty} \frac{1}{t} \log M_t = E\left(\log M_{t+1} - \log M_t\right) \leq 0.$$

Suppose now we consider expectations under the probabilities implied by the multiplicative martingale $\{M_t : t \geq 0\}$. Then the second inequality in (7.1) implies that

$$\lim_{t \to \infty} \frac{1}{t} \log M_t = \widetilde{E}\left(\log M_{t+1} - \log M_t\right) \geq 0.$$

These two limits give a large sample justification for use of the criterion

$$\lim_{t \to \infty} \frac{1}{t} \log M_t = \lim_{t \to \infty} \left[\frac{1}{t} \sum_{\tau=1}^{t} \left(\log M_\tau - \log M_{\tau-1}\right) + \frac{1}{t} \log M_0\right]$$

to determine which of the two models generates the data since the inequalities are strict when the multiplicative martingale is not degenerate.

Proposition 7.2

The process $\{\log M_t : t \geq 0\}$ is a **supermartingale**. That is,

$$E\left(\log M_{t+1} \mid X_t, M_t\right) \leq \log M_t, \quad t \geq 0.$$

The process $\{M_t \log M_t : t \geq 0\}$ is a **submartingale**. That is,

$$E\left(M_{t+1} \log M_{t+1} \mid X_t, M_t\right) \geq M_t \log M_t, \quad t \geq 0.$$

These insights extend directly to the case of multiple models captured by alternative multiplicative martingales. It leads to the use of

$$\max_{i=0,1,\ldots,\ell} \frac{1}{t} \log M_t^i$$

as a way to select models with a large sample justification. As we will show, this approach has a direct application to the method of maximum likelihood.

We conclude this section by pointing to an unusual sample path property of multiplicative martingales.

Proposition 7.3

For a nondegenerate multiplicative martingale, $\{M_t : t \geq 0\}$ converges almost surely to zero even though $E\left(\frac{M_t}{M_0} \mid X_0\right) = 1$ for each t.

Proof. We observed that by the Law of Large Numbers:

$$\lim_{t\to\infty} \frac{1}{t} \log M_t$$

converges to a negative limit almost surely. This in turn implies that

$$\lim_{t\to\infty} M_t = 0$$

almost surely.

[Martin, 2012] uses this type of result to argue that cumulative returns on long-dated assets that are positive martingales necessarily have distributions with fat tails. Section Illustrating the decompositions: a worked example of Chapter 8 quantifies this behavior for a log-normal example.

Here are some examples of multiplicative martingales constructed from some standard probability models.

Example 7.1

Consider a baseline Markov process having transition probability density π_o with respect to a measure λ over the state space $\mathcal{X}$

$$P_o(dx^+|x) = \pi_o(x^+ \mid x)\lambda(dx^+)$$

Let π denote some other transition density that we represent as

$$\pi(x^+ \mid x)\lambda(dx^+) = \left[\frac{\pi(x^+ \mid x)}{\pi_o(x^+ \mid x)}\right]\pi_o(x^+ \mid x)\lambda(dx^+)$$

where we assume that $\pi_o(x^+ \mid x) = 0$ implies that $\pi(x^+ \mid x) = 0$ for all x^+ and x in $\mathcal{X}$. Construct the multiplicative increment process as:

$$N_{t+1} = \frac{\pi(X_{t+1} \mid X_t)}{\pi_o(X_{t+1} \mid X_t)}.$$

Example 7.2

Let an alternative model for a vector X be a vector autoregression:

$$X_{t+1} = \mathbb{A}X_t + \mathbb{B}W_{t+1}$$

where $\mathbb{A}$ is a stable matrix, $\{W_{t+1} : t \geq 0\}$ is an i.i.d. sequence of $\mathcal{N}(0, I)$ random vectors conditioned on X_0, and $\mathbb{B}$ is a square, nonsingular matrix. Assume that a baseline model for X has the same functional form but different settings $(\mathbb{A}_o, \mathbb{B}_o)$ of its parameters. Construct N_{t+1} as the one-period conditional log-likelihood ratio

$$\begin{aligned}
\log N_{t+1} = &-\frac{1}{2}(X_{t+1} - \mathbb{A}X_t)'(\mathbb{B}\mathbb{B}')^{-1}(X_{t+1} - \mathbb{A}X_t) \\
&+\frac{1}{2}(X_{t+1} - \mathbb{A}_oX_t)'(\mathbb{B}_o\mathbb{B}_o{}')^{-1}(X_{t+1} - \mathbb{A}_oX_t) \\
&-\frac{1}{2}\log\det(\mathbb{B}\mathbb{B}') + \frac{1}{2}\log\det(\mathbb{B}_o\mathbb{B}_o{}')
\end{aligned}$$

Notice how the matrices $(\mathbb{A}_o, \mathbb{B}_o)$ of the baseline model and parameters $(\mathbb{A}, \mathbb{B})$ of the alternative model both appear.

Remark 7.2

Because $\mathbb{B}$ is a nonsingular square matrix, model Example 7.2 has the same number of shocks, i.e., entries of W, as there are components of X. A more general setting would be a hidden Markov state model like the one presented in Section Kalman Filter and Smoother of Chapter 6 that has a time-invariant representation with an "information state vector" constructed as a way to condition on an infinite past of an observation vector.

7.5. Log likelihoods

We now construct the function κ as the increment to a log-likelihood ratio. We let θ denote a parameter vector, and suppose that a vector of date $t+1$ observations Z_{t+1} is given by

$$Z_{t+1} = \phi(X_t, W_{t+1}, \theta).$$

Form:

$$\begin{aligned}\kappa(x, w^* \mid \theta) &= \log \psi(z^* \mid x, \theta) - \log \psi(z^* \mid x, \theta_0) \\ &= \log \psi\left[\phi(x, w^*, \theta) \mid x, \theta\right] - \log \psi\left[\phi(x, w^*, \theta_0) \mid x, \theta_0\right]\end{aligned}$$

where $\psi(z^* \mid x, \theta)$ is the density of Z_{t+1} given X_t and θ. We further assume that given (x, θ), Z_{t+1} is informationally equivalent to W_{t+1}. This can be justified by an assumption that we can invert ϕ as a function of W_{t+1}. We assume that given θ, X_t is revealed by past Z_t and possibly a finite number of lags. Model θ_0 is the baseline used for computing expectations.

By factoring a joint density we obtain a log-likelihood process conditioned on (X_0, θ) as:

$$L_t(\theta) = \sum_{j=1}^{t} \log \psi(Z_j \mid X_{j-1}, \theta).$$

Chapter 6 writes $L_t(\theta)$ for the likelihood itself; here and in what follows $L_t(\theta)$ is its logarithm. We apply our previous arguments to justify finding the

$$\max_{i=0,1,\ldots,\ell} L_t(\theta_i)$$

for a discrete set of models.[1] In what follows, we suppose there is a continuum of models.

7.6. Score processes

In this section, we study first-order necessary conditions associated with the maximum likelihood estimator. For simplicity suppose that the parameter space Θ is an open interval of $\mathbb{R}$ containing the parameter value θ_0. Inequality (7.1) implies that the parameter value θ_0 necessarily maximizes the population objective

$$\int_{\mathcal{Z}} \log \psi(z^*|x, \theta)\psi(z^*|x, \theta_0)\tau(dz^*). \tag{7.2}$$

Importantly, this objective conditions on x and θ_0. Suppose that we can differentiate under the integral sign in (7.2) to get the first-order condition

$$\int_{\mathcal{Z}} \left[\frac{d}{d\theta} \log \psi(z^*|x, \theta_0)\right] \psi(z^*|x, \theta_0)\tau(dz^*) = 0. \tag{7.3}$$

Definition 7.2

The **score process** $\{S_t : t = 0, 1, \ldots\}$ is defined as

$$S_t = \frac{d}{d\theta} L_t(\theta)\bigg|_{\theta_0} = \sum_{j=1}^{t} \frac{d}{d\theta} \log \psi(Z_j | X_{j-1}, \theta_0).$$

Theorem 7.1

$E\left(S_{t+1} - S_t \mid X_t\right) = 0$, so the score process is an additive martingale with increment $\frac{d}{d\theta}\log\psi(Z_j|X_{j-1}, \theta_0)$ under the θ_0 probability model. From the analysis in Chapter 3 and [Billingsley, 1961], we obtain a central limit approximation:

$$\frac{1}{\sqrt{t}} S_t \Rightarrow \mathcal{N}(0, V)$$

where $V = E\left(\left[\frac{d}{d\theta}\log\psi(Z_{t+1}|X_t, \theta_0)\right]^2\right)$.

Proof. This follows directly from equation (7.3) and Proposition 3.2.

Theorem 7.1 justifies using the martingale central limit theorem Proposition 3.2 to characterize the large sample behavior of the score process. The resulting central limit approximation yields a large sample characterization of the maximum likelihood estimator of θ in a Markov setting. With additional regularity conditions, the following result is typical:

$$\sqrt{t}(\theta_t - \theta_0) \Rightarrow \mathcal{N}\left(0, \frac{1}{V}\right),$$

where θ_t maximizes the log likelihood function $L_t(\theta)$. This kind of result motivates interpreting the covariance of the martingale increment of the score process

$$V = E\left(\left[\frac{d}{d\theta}\log\psi(Z_{t+1}|X_t, \theta_0)\right]^2\right)$$

as a measure of the information in the data about the parameter θ_0. The scalar V is a measure of what is called **Fisher information** after the statistician R.A. Fisher. This analysis has a direct extension to the case where θ is a vector.

Consider extending the notion of Fisher information to the following situation. There is a vector θ of unknown parameters. We want a measure of information about one component of the parameter vector, although to get that information we have to estimate other components too.

Call the first block the ‘parameters of interest’ and write θ for it; call the second block the ‘nuisance parameters’ and write ϑ for it. Suppose that the likelihood is parameterized on an open set Θ in a finite dimensional Euclidean space and that the true parameter vector is $(\theta_0, \vartheta_0) \in \Theta$. Write the multivariate score process as

$$\left\{ \begin{bmatrix} S_{t+1} \\ \tilde{S}_{t+1} \end{bmatrix} : t = 0, 1, \ldots \right\}$$

where $\{S_{t+1} : t = 0, 1, \ldots\}$ is the partial derivative of the log-likelihood with respect to θ and $\{\tilde{S}_{t+1} : t = 0, 1, \ldots\}$ is the partial derivative with respect to ϑ. Estimating ϑ_0 simultaneously with θ_0 is more difficult than estimating θ_0 when knowing ϑ_0. Fisher’s measure of information corrects for the additional challenge of estimating ϑ_0 simultaneously in order to make inferences about θ_0. In particular, Fisher’s measure of information about θ is the inverse of the $(1, 1)$ component of the inverse of the covariance matrix $\mathbb{V}$, partitioned conformably with θ, ϑ:

$$\mathbb{V} = \begin{bmatrix} E(S_{t+1} - S_t)^2 & E(S_{t+1} - S_t)(\tilde{S}_{t+1} - \tilde{S}_t)' \\ E(\tilde{S}_{t+1} - \tilde{S}_t)(S_{t+1} - S_t) & E(\tilde{S}_{t+1} - \tilde{S}_t)(\tilde{S}_{t+1} - \tilde{S}_t)' \end{bmatrix}.$$

To represent this inverse in a revealing way, compute the population regression

$$S_{t+1} - S_t = \beta'(\tilde{S}_{t+1} - \tilde{S}_t) + U_{t+1},$$

where β is the population regression coefficient and U_{t+1} is the population regression residual that by construction is orthogonal to the regressor $(\tilde{S}_{t+1} - \tilde{S}_t)$. The population regression induces the representation

$$\begin{bmatrix} S_{t+1} - S_t \\ \tilde{S}_{t+1} - \tilde{S}_t \end{bmatrix} = \begin{bmatrix} 1 & \beta' \\ \mathbb{O} & \mathbb{I} \end{bmatrix} \begin{bmatrix} U_{t+1} \\ \tilde{S}_{t+1} - \tilde{S}_t \end{bmatrix},$$

where $\mathbb{O}$ is a vector of zeros. Since U_{t+1} is orthogonal to $\tilde{S}_{t+1} - \tilde{S}_t$,

$$\mathbb{V} = \begin{bmatrix} 1 & \beta' \\ \mathbb{O} & \mathbb{I} \end{bmatrix} \begin{bmatrix} EU_{t+1}^2 & \mathbb{O}' \\ \mathbb{O} & E(\tilde{S}_{t+1} - \tilde{S}_t)(\tilde{S}_{t+1} - \tilde{S}_t)' \end{bmatrix} \begin{bmatrix} 1 & \mathbb{O}' \\ \beta & \mathbb{I} \end{bmatrix}.$$

The inverse matrix is:

$$\mathbb{V}^{-1} = \begin{bmatrix} 1 & \mathbb{O}' \\ -\beta & \mathbb{I} \end{bmatrix} \begin{bmatrix} \frac{1}{EU_{t+1}^2} & \mathbb{O}' \\ \mathbb{O} & \left[E(\tilde{S}_{t+1} - \tilde{S}_t)(\tilde{S}_{t+1} - \tilde{S}_t)'\right]^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\beta' \\ \mathbb{O} & \mathbb{I} \end{bmatrix}.$$

This equation reveals that the $(1,1)$ component of the partition of matrix $\mathbb{V}^{-1}$ is

$$\mathbb{V}_{1,1}^{-1} = \frac{1}{E(U_{t+1}^2)}.$$

We therefore take the Fisher information about θ to be $E(U_{t+1}^2)$, the reciprocal of $\mathbb{V}_{1,1}^{-1}$. By least squares theory, this measure satisfies

$$E\left(U_{t+1}^2\right) \leq E\left[(S_{t+1} - S_t)^2\right]. \tag{7.4}$$

This inequality asserts that the likelihood function contains more information about θ when ϑ is known to be ϑ_0 than when θ and ϑ are both unknown.

Inequality (7.4) offers reasons to be cautious when ignoring the joint estimation challenge and pretending you know ϑ. This latter practice leads you to overstate the information in the sample about the parameter θ of interest. Note that since the multivariate score $\begin{bmatrix} S_{t+1} \\ \tilde{S}_{t+1} \end{bmatrix}$ is a vector of additive martingales, the score regression residual U_{t+1} is a martingale difference, so its partial sums form an additive martingale.

7.7. Using a multiplicative martingale for model selection

We now return to the single multiplicative martingale as a way to model an alternative probability. Suppose we use the martingale at time t, M_t, to construct a criterion for choosing between models. One choice is to check whether M_t is greater than or less than one for a sample of size t. When M_t exceeds one, we choose the probability model induced by the multiplicative martingale. For motivation, recall from Proposition 7.3 that the martingale converges almost surely to zero. Thus the probability of exceeding a fixed threshold must decline to zero under the baseline specification. To provide a more refined result, we study probabilities of making mistakes when using this criterion. We follow [Chernoff, 1952] and others by applying what is called large deviation theory to study the probability of making a mistake if the baseline model used to represent expectations governs the data generation. We show how to characterize this probability for large sample sizes.

For notational convenience, we initialize $M_0 = 1$ and condition on the date zero state X_0. Should the $\{M_t : t \geq 0\}$ of interest be initialized differently, in what follows we would use M_t/M_0 and include M_0 in the date zero conditioning information. Instead of carrying along the extra notation, we just change how we normalize the martingale. We construct a bound on the mistake probability by using an inequality that is implied by two facts:

- The probability that M_t exceeds one is the expectation of the indicator function:

$$\mathbf{1}_{[1,\infty)}(m) \stackrel{\text{def}}{=} \begin{cases} 0 & m < 1 \\ 1 & m \geq 1 \end{cases}$$

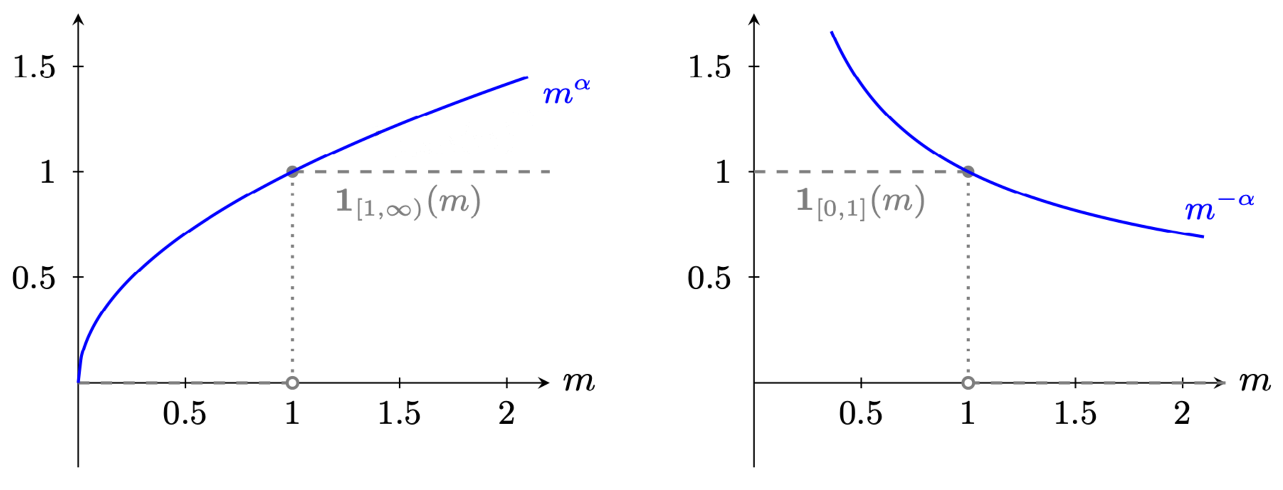
- For a scalar $\alpha > 0$,

$$\mathbf{1}_{[1,\infty)}(m) \leq m^{\alpha}.$$

The left panel of Fig. 7.2 illustrates this inequality. Values of $0 < \alpha < 1$ are of particular interest in which case m^{α} is a concave increasing function. Thus we use the inequality

$$E\left[\mathbf{1}_{[1,\infty)}(M_t) \mid X_0\right] \leq E\left[(M_t)^{\alpha} \mid X_0\right].$$

The expectation on the right side of this relation is more tractable to analyze than the one on the left side when we investigate behavior as $t \to \infty$. Since the inequality applies for arbitrary α, to produce approximations that are as sharp as possible, we will minimize over the choice of α.



(a) The inequality $\mathbf{1}_{[1,\infty)}(m) \leq m^{\alpha}$. (b) The inequality $\mathbf{1}_{[0,1]}(m) \leq m^{-\alpha}$.

Fig. 7.2 Two indicator function inequalities with $\alpha = \frac{1}{2}$.

Note that m^{α} for $0 < \alpha < 1$ is a concave function. From the associated gradient inequality, the process $\{(M_t)^{\alpha} : t \geq 0\}$ is a supermartingale. This uses the same logic as we used when showing that $\{\log M_t : t \geq 0\}$ is a supermartingale. The limit of interest is

$$-\lim_{t\to\infty} \frac{1}{t} \log E\left[(M_t)^{\alpha} \mid X_0\right] \stackrel{\text{def}}{=} \eta(\alpha).$$

The resulting $\eta(\alpha)$ is interpretable as an **asymptotic decay rate** for the process $\{(M_t)^\alpha : t \geq 0\}$. To obtain a sharp bound on the limiting behavior of the mistake probability we solve

$$\eta^* \stackrel{\text{def}}{=} \max_{0<\alpha<1} \eta(\alpha).$$

Then η^* tells us the asymptotic decay rate in the probability of choosing the multiplicative martingale-based model when the original baseline model actually generates the data.

So far, we have only analyzed one type of mistake. The other possibility is that the alternative model with probabilities induced by the multiplicative martingale generates the data, and $M_t < 1$. In this case, the indicator function and the dominating function are given by:

$$\mathbf{1}_{[0,1]}(m) \leq m^{-\alpha},$$

as is illustrated in the right panel of Fig. 7.2. Since we are interested in computations under the alternative probability measure, we are led to study

$$-\lim_{t\to\infty} \frac{1}{t} \log E\left[(M_t)^{1-\alpha} \mid X_0\right] \stackrel{\text{def}}{=} \tilde{\eta}(\alpha)$$

Again, we maximize over α and note that

$$\max_{0<\alpha<1} \tilde{\eta}(\alpha) = \max_{0<\alpha<1} \eta(\alpha) \stackrel{\text{def}}{=} \eta^*$$

The asymptotic decay rates of the two mistakes are *equated* when one is used as the threshold. The resulting decay rate is called **Chernoff entropy**. This finding does not imply that the two mistake probabilities are equated. But the rate equalization extends to constant thresholds other than one, including a Bayesian procedure with prior probabilities assigned to each model. Alternative constant thresholds will change the probabilities but not the asymptotic decay rates. Commonly employed classical methods that hold fixed one of the error probabilities independent of sample size do not have fixed thresholds and thus are not covered by this analysis. Nor are they well justified. In contrast to this common approach, fixed threshold rules have both mistake probabilities decay as more data become available.

In Chapter 8, we develop methods that, among other things, show how to compute the asymptotic decay rates.

Remark 7.3

Consider a Bayesian decision maker selecting a model. Let π_0 denote the prior probability that the baseline model generates the data, and let π_m denote the prior probability that the alternative model implied by the multiplicative martingale generates the data. Let υ_0 denote the utility reward if the baseline model is chosen correctly, and let υ_m denote the utility reward if the multiplicative martingale model is chosen correctly. At time t, the decision maker will select the baseline model by checking if the utility-weighted posterior probability of the baseline model exceeds the multiplicative model counterpart:

$$\upsilon_m \frac{\pi_m M_t}{\pi_0 + \pi_m M_t} \leq \upsilon_0 \frac{\pi_0}{\pi_0 + \pi_m M_t}$$

With a straightforward computation, this inequality simplifies to:

$$M_t \leq \frac{\upsilon_0 \pi_0}{\upsilon_m \pi_m},$$

which is a constant threshold rule of the type we have been considering.

7.8. Summary

A multiplicative martingale induces an alternative probability specification, and a likelihood ratio process is such a martingale. That is what allows a family of competing models to be represented as a family of martingales over a common baseline. The logarithm of each is an additive functional in the sense of Chapter 4 with a negative trend coefficient, which is why Proposition 7.3 holds: data eventually discriminate among the models without any prior over them. Chernoff entropy measures how fast.

Score processes come from differentiating a log likelihood with respect to a parameter, and are additive martingales. Their long-run variance is the Fisher information that limits what can be learned about the parameter, and inequality (7.4) records what is lost when nuisance parameters are unknown. Chapter 8 studies multiplicative functionals without imposing the martingale restriction, and Chapter 13 turns to estimation when no likelihood is specified at all.

7.9. Exercises

Exercise 7.1 (A Gaussian likelihood ratio and its Chernoff entropy)

Model 0 asserts that a scalar observation Z_t is drawn i.i.d. from $\mathcal{N}(\mu_0, \sigma^2)$ and model 1 asserts that it is drawn i.i.d. from $\mathcal{N}(\mu_1, \sigma^2)$, with $\mu_0 \neq \mu_1$ and $\sigma^2 > 0$ known. Define the log-likelihood ratio increment

$$\kappa(Z_{t+1}) = \log \frac{\psi(Z_{t+1} \mid \theta_1)}{\psi(Z_{t+1} \mid \theta_0)}$$

and the cumulated ratio $\log M_t = \sum_{\tau=1}^{t} \kappa(Z_\tau)$, and write $\delta = (\mu_1 - \mu_0)/\sigma$ for the signal-to-noise ratio.

(a) Show that $\kappa(Z_{t+1}) = \frac{\mu_1-\mu_0}{\sigma^2} Z_{t+1} - \frac{\mu_1^2-\mu_0^2}{2\sigma^2}$, and that under model 0 it is distributed $\mathcal{N}\left(-\frac{\delta^2}{2}, \delta^2\right)$.

(b) Verify directly that $\{M_t\}$ is a multiplicative martingale under model 0 in the sense of Definition 7.1, and confirm both inequalities of (7.1).

(c) Compute the asymptotic decay rate $\eta(\alpha)$ for $\alpha \in (0,1)$, find the maximizing α^*, and report the Chernoff entropy η^*. Verify the symmetry $\eta(1-\alpha) = \eta(\alpha)$ asserted in the chapter.

(d) How does η^* depend on δ, and what happens as $\mu_1 \to \mu_0$? Use Remark 7.3 to say what a Bayesian who assigns prior probabilities π_0, π_m would do differently, and whether it changes η^*.

Exercise 7.2 (Fisher information in the presence of a nuisance parameter)

Take the same i.i.d. Gaussian setting, but now treat the parameters as unknown. The single-observation log-likelihood is

$$\log \psi\left(z \mid \mu, \sigma^2\right) = -\frac{1}{2} \log\left(2\pi\sigma^2\right) - \frac{(z-\mu)^2}{2\sigma^2},$$

and the true values are $\mu_0 = 0$, $\sigma_0^2 = 1$.

(a) Compute the score increment $\frac{d}{d\mu} \log \psi\left(Z_t \mid \mu_0\right)$ holding σ^2 known, and verify that its expectation under the true model is zero, confirming (7.3).

(b) Compute the Fisher information V from the score, then verify the *information equality* by computing $-\mathbb{E}_0\left[\frac{d^2}{d\mu^2} \log \psi\left(Z_t \mid 0\right)\right]$ and confirming that both give the same value.

(c) Now let $\vartheta = \sigma^2$ be an unknown nuisance parameter alongside the parameter of interest $\theta = \mu$. Compute the 2×2 matrix $\mathbb{V}$, obtain the population regression coefficient β of the score for μ on the score for σ^2, and compute the residual variance $\mathbb{E}\left(U_{t+1}^2\right)$.

(d) Does inequality (7.4) hold strictly here? Explain which feature of the Gaussian family accounts for this, and give an informal argument for why the inequality would be strict in a family whose mean and variance share a parameter.

Exercise 7.3 (A likelihood ratio for two autoregressive models)

Consider two models for a scalar observation,

$$Z_{t+1} = \mathsf{n}_0 + (\mathsf{d}_0)^\top X_t + \mathsf{f}_0 W_{t+1}$$
$$Z_{t+1} = \mathsf{n}_1 + (\mathsf{d}_1)^\top X_t + \mathsf{f}_1 W_{t+1},$$

where W_{t+1} is a scalar standard normal shock, $\mathsf{f}_0, \mathsf{f}_1 > 0$, and X_t consists of Z_t and finitely many lags. Each model specifies coefficients $\theta_j = (\mathsf{n}_j, \mathsf{d}_j, \mathsf{f}_j)$ with $\theta_1 \neq \theta_0$. Suppose model θ_0 generates the data.

(a) Use the date $t+1$ contribution to the log-likelihood ratio of model 1 relative to model 0 to form $\kappa(X_t, W_{t+1})$, and show that it can be written as $\mathsf{a}_t + \mathsf{b}_t W_{t+1} + \frac{\mathsf{c}}{2} W_{t+1}^2$. Identify a_t, b_t, and c, and note which of them depend on X_t.

(b) Verify that $\mathbb{E}\left(\exp\left[\kappa(X_t, W_{t+1})\right] \mid X_t, \theta_0\right) = 1$, so the multiplicative functional is a martingale.

(c) Verify that $\mathbb{E}\left(\kappa(X_t, W_{t+1}) \mid X_t, \theta_0\right) < 0$, and identify the two separate sources of the strict inequality: one from the conditional means differing, one from the volatilities differing.

(d) Show that for $0 < \alpha < 1$, $\mathbb{E}\left(\exp\left[\alpha\kappa(X_t, W_{t+1})\right] \mid X_t, \theta_0\right) < 1$, and state the restriction on $\mathsf{f}_0/\mathsf{f}_1$ needed for the expectation to be finite.

In checking these relations you may use the following. If W is standard normal then

$$\mathbb{E}\left[\exp\left(\mathsf{a} + \mathsf{b}W + \frac{\mathsf{c}}{2}W^2\right)\right] = \frac{1}{\sqrt{1-\mathsf{c}}}\exp\left[\mathsf{a} + \frac{\mathsf{b}^2}{2(1-\mathsf{c})}\right]$$

for $\mathsf{c} < 1$. You do not have to derive it, although a complete-the-square argument does the job.

Exercise 7.4 (Computing a Chernoff entropy)

Continue with Exercise 7.3, but now suppose $\mathsf{d}_0 = \mathsf{d}_1 = 0$, so that X_t drops out of your formulas and the observations are i.i.d. Chapter 8 develops general methods for computing decay rates; this special case can be done by hand.

(a) Form $\eta(\alpha) = -\log\mathbb{E}\left(\exp\left[\alpha\kappa\right] \mid \theta_0\right)$ as an explicit function of α and maximize over $0 < \alpha < 1$. You may do the maximization analytically, or numerically for a few parameter configurations.

(b) Specialize to $\mathsf{f}_0 = \mathsf{f}_1 = \mathsf{f}$. Show that $\alpha^* = \frac{1}{2}$ and that the Chernoff entropy reduces to $\left(\mathsf{n}_0 - \mathsf{n}_1\right)^2 / \left(8\mathsf{f}^2\right)$, recovering Exercise 7.1.

(c) Discuss the effect of raising f_0 and f_1 together while holding n_0 and n_1 fixed, and then of widening the gap between n_0 and n_1 while holding $\mathsf{f}_0 = \mathsf{f}_1$.

(d) Now let $\mathsf{f}_0 \neq \mathsf{f}_1$ with $\mathsf{n}_0 = \mathsf{n}_1$, so the models differ *only* in volatility. Show that the Chernoff entropy is still strictly positive, and that $\alpha^* \neq \frac{1}{2}$ in general. What does the asymmetry of $\eta(\cdot)$ about $\frac{1}{2}$ tell you that the equal-volatility case conceals?

7.10. Answers

Solution to Exercise 7.1 (A Gaussian likelihood ratio and its Chernoff entropy)

Click to hide

(a) The σ^2 terms cancel in the ratio, leaving

$$\kappa = \frac{(Z_{t+1} - \mu_0)^2 - (Z_{t+1} - \mu_1)^2}{2\sigma^2} = \frac{\mu_1 - \mu_0}{\sigma^2} Z_{t+1} - \frac{\mu_1^2 - \mu_0^2}{2\sigma^2}.$$

Writing $Z_{t+1} = \mu_0 + \sigma W_{t+1}$ under model 0 and collecting terms,

$$\kappa = \frac{\mu_1 - \mu_0}{\sigma^2}\left(\mu_0 - \frac{\mu_1 + \mu_0}{2}\right) + \frac{\mu_1 - \mu_0}{\sigma} W_{t+1} = -\frac{\delta^2}{2} + \delta W_{t+1},$$

which is $\mathcal{N}\left(-\delta^2/2, \delta^2\right)$.

(b) By the log-normal formula, $\mathbb{E}_0\left[\exp \kappa\right] = \exp\left(-\frac{\delta^2}{2} + \frac{\delta^2}{2}\right) = 1$, which is Definition 7.1. For (7.1), $\mathbb{E}_0(\kappa) = -\delta^2/2 < 0$ confirms the first inequality. For the second, under the alternative measure κ has mean $+\delta^2/2$, since tilting by M shifts the mean of W from 0 to δ; so $\mathbb{E}_0\left[\kappa \exp \kappa\right] = \widetilde{\mathbb{E}}(\kappa) = +\delta^2/2 > 0$. The two inequalities are strict, as they must be whenever $\delta \neq 0$.

(c) Since the κ's are i.i.d.,

$$\mathbb{E}_0\left[(M_t)^\alpha\right] = \left(\mathbb{E}_0\left[e^{\alpha\kappa}\right]\right)^t = \exp\left(\frac{(\alpha^2 - \alpha)\delta^2}{2}\, t\right),$$

so

$$\eta(\alpha) = -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \left[(M_t)^\alpha \right] = \frac{\alpha(1-\alpha)\delta^2}{2},$$

a downward parabola. Setting $\eta'(\alpha) = (1-2\alpha)\delta^2/2 = 0$ gives $\alpha^* = \frac{1}{2}$ and

$$\eta^* = \frac{\delta^2}{8} = \frac{(\mu_1 - \mu_0)^2}{8\sigma^2}.$$

Symmetry is immediate from the formula: $\eta(1-\alpha) = \frac{(1-\alpha)\alpha\delta^2}{2} = \eta(\alpha)$. This is the concrete instance of the general claim in the chapter that the two mistake probabilities decay at a common rate.

(d) η^* grows with the squared signal-to-noise ratio. As $\mu_1 \to \mu_0$ it goes to zero: the models become statistically indistinguishable, and the number of observations needed to separate them grows without bound. Small belief distortions can therefore survive statistical scrutiny, a theme taken up in Chapter 8.

By Remark 7.3, a Bayesian with priors π_0, π_m and utility rewards υ_0, υ_m compares M_t to the *constant* threshold $\frac{\upsilon_0 \pi_0}{\upsilon_m \pi_m}$ rather than to one. Changing the threshold changes both mistake probabilities in finite samples, but not their exponential decay rates: a constant shifts $\log M_t$ by a fixed amount while $\log M_t$ itself drifts linearly in t. So η^* is unchanged. The chapter's warning applies here: a classical procedure that holds one error probability fixed as the sample grows does *not* use a fixed threshold and is not covered by this analysis.

Solution to Exercise 7.2 (Fisher information in the presence of a nuisance parameter)

Click to hide

(a) With $\sigma^2 = 1$, $\frac{d}{d\mu} \log \psi(Z_t \mid \mu) = Z_t - \mu$, which at $\mu_0 = 0$ is Z_t. Under the true model $\mathbb{E}(Z_t) = 0$, confirming (7.3).

(b) From the score, $V = \mathbb{E}\left(Z_t^2\right) = 1$. From the second derivative, $\frac{d^2}{d\mu^2} \log \psi = -1$, so $-\mathbb{E}_0 \left[\frac{d^2}{d\mu^2} \log \psi \right] = 1$. Both give $V = 1$.

(c) The two score increments at $\left(\mu_0, \sigma_0^2\right) = (0, 1)$ are

$$S_{t+1} - S_t = \frac{\partial}{\partial \mu} \log \psi = Z_{t+1}, \qquad \tilde{S}_{t+1} - \tilde{S}_t = \frac{\partial}{\partial \sigma^2} \log \psi = -\frac{1}{2} + \frac{Z_{t+1}^2}{2}.$$

Under $\mathcal{N}(0,1)$ we have $\mathbb{E}Z^2 = 1$, $\mathbb{E}Z^4 = 3$, and all odd moments vanish. Hence

$$\mathbb{E}\left[Z\left(-\frac{1}{2}+\frac{Z^2}{2}\right)\right]=0, \qquad \mathbb{E}\left[\left(-\frac{1}{2}+\frac{Z^2}{2}\right)^2\right]=\frac{1}{4}-\frac{1}{2}+\frac{3}{4}=\frac{1}{2},$$

so

$$\mathbb{V}=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{2}\end{bmatrix}.$$

The off-diagonal zero gives $\beta=0$, hence $U_{t+1}=Z_{t+1}$ and $\mathbb{E}\left(U_{t+1}^2\right)=1=\mathbb{E}\left[\left(S_{t+1}-S_t\right)^2\right]$.

(d) No: (7.4) holds with *equality*. The responsible feature is that the two scores are orthogonal, which for the Gaussian family is the population counterpart of the independence of the sample mean and the sample variance. Not knowing σ^2 costs nothing when the object of interest is μ.

The Fisher information about θ is the squared length of the *component of the θ-score orthogonal to the ϑ-score*. Whenever the two scores are correlated, that projection strictly shortens the vector and the inequality is strict. Suppose instead that a single parameter governed both moments, say $Z_t \sim \mathcal{N}\left(\theta, \theta^2\right)$. Then perturbing θ moves the mean and the spread together, a change in location can be partly mimicked by a change in scale, and the two directions of the likelihood surface are no longer orthogonal. The data can then say less about location alone than they could if the scale were known, the caution the chapter attaches to inequality (7.4).

Solution to Exercise 7.3 (A likelihood ratio for two autoregressive models)

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(a) The conditional density of Z_{t+1} given X_t under model j is normal with mean $\mathsf{n}_j+\left(\mathsf{d}_j\right)^\top X_t$ and variance f_j^2. Under model 0 the realized observation satisfies $Z_{t+1}-\mathsf{n}_0-\left(\mathsf{d}_0\right)^\top X_t=\mathsf{f}_0 W_{t+1}$, so

$$Z_{t+1}-\mathsf{n}_1-\left(\mathsf{d}_1\right)^\top X_t=\mathsf{m}_t+\mathsf{f}_0 W_{t+1}, \qquad \mathsf{m}_t \stackrel{\text{def}}{=}\left(\mathsf{n}_0-\mathsf{n}_1\right)+\left(\mathsf{d}_0-\mathsf{d}_1\right)^\top X_t.$$

Therefore

$$\kappa=\log\frac{\mathsf{f}_0}{\mathsf{f}_1}-\frac{\left(\mathsf{m}_t+\mathsf{f}_0 W_{t+1}\right)^2}{2\mathsf{f}_1^2}+\frac{W_{t+1}^2}{2},$$

which is of the stated form with

$$\mathsf{a}_t = \log \frac{\mathsf{f}_0}{\mathsf{f}_1} - \frac{\mathsf{m}_t^2}{2\mathsf{f}_1^2}, \qquad \mathsf{b}_t = -\frac{\mathsf{m}_t \mathsf{f}_0}{\mathsf{f}_1^2}, \qquad \mathsf{c} = 1 - \frac{\mathsf{f}_0^2}{\mathsf{f}_1^2}.$$

Only a_t and b_t depend on the state, and they do so only through m_t, the gap between the two conditional means. The quadratic coefficient c is a constant reflecting the volatility discrepancy alone.

(b) Here $1 - \mathsf{c} = \mathsf{f}_0^2/\mathsf{f}_1^2 > 0$, so the formula applies:

$$\mathbb{E}\left[e^{\kappa} \mid X_t\right] = \frac{\mathsf{f}_1}{\mathsf{f}_0} \exp\left[\log \frac{\mathsf{f}_0}{\mathsf{f}_1} - \frac{\mathsf{m}_t^2}{2\mathsf{f}_1^2} + \frac{\mathsf{m}_t^2 \mathsf{f}_0^2/\mathsf{f}_1^4}{2\mathsf{f}_0^2/\mathsf{f}_1^2}\right] = \frac{\mathsf{f}_1}{\mathsf{f}_0} \cdot \frac{\mathsf{f}_0}{\mathsf{f}_1} = 1.$$

The two m_t^2 terms cancel, which makes the ratio a martingale for *every* state.

(c) Since $\mathbb{E}W = 0$ and $\mathbb{E}W^2 = 1$, $\mathbb{E}\left[\kappa \mid X_t\right] = \mathsf{a}_t + \mathsf{c}/2$. Writing $\mathsf{r} = \mathsf{f}_0^2/\mathsf{f}_1^2$,

$$\mathbb{E}\left[\kappa \mid X_t\right] = \frac{1}{2}\left[\log \mathsf{r} + 1 - \mathsf{r}\right] - \frac{\mathsf{m}_t^2}{2\mathsf{f}_1^2}.$$

Both terms are non-positive. The first vanishes only at $\mathsf{r} = 1$, by strict concavity of the logarithm at 1; it is the contribution of the *volatility* discrepancy. The second vanishes only when $\mathsf{m}_t = 0$; it is the contribution of the *conditional mean* discrepancy. Two models can therefore be distinguished either because they disagree about where Z_{t+1} will be or because they disagree about how uncertain it is, and the log-likelihood drifts down under either kind of disagreement.

(d) Applying the formula with $\alpha\kappa = \alpha\mathsf{a}_t + \alpha\mathsf{b}_t W + \frac{\alpha\mathsf{c}}{2} W^2$,

$$\mathbb{E}\left[e^{\alpha\kappa} \mid X_t\right] = \frac{1}{\sqrt{1 - \alpha\mathsf{c}}} \exp\left[\alpha\mathsf{a}_t + \frac{\alpha^2 \mathsf{b}_t^2}{2\left(1 - \alpha\mathsf{c}\right)}\right],$$

which is finite provided $\alpha\mathsf{c} < 1$. Since $\mathsf{c} = 1 - \mathsf{f}_0^2/\mathsf{f}_1^2 < 1$ always, and $\alpha \in (0, 1)$, the requirement $\alpha\mathsf{c} < 1$ is automatic. That the expectation is strictly less than one is the gradient inequality of the chapter: m^α is strictly concave on $(0, \infty)$ for $\alpha \in (0, 1)$, so by Jensen's inequality $\mathbb{E}\left[(e^\kappa)^\alpha\right] < \left(\mathbb{E}\left[e^\kappa\right]\right)^\alpha = 1$, using (b). This is the inequality that underlies the fixed-threshold analysis of the chapter and delivers a strictly positive decay rate.

Solution to [Exercise 7.4 (Computing a Chernoff entropy)](#)

Click to hide

(a) With $\mathsf{d}_0 = \mathsf{d}_1 = 0$, $\mathsf{m}_t = \mathsf{m} \overset{\text{def}}{=} \mathsf{n}_0 - \mathsf{n}_1$ is constant, so a and b no longer depend on the state and the κ's are i.i.d. Taking the negative logarithm of the expression in part (d) of Exercise 7.3,

$$\eta(\alpha) = \frac{1}{2}\log\left(1 - \alpha\mathsf{c}\right) - \alpha\mathsf{a} - \frac{\alpha^2\mathsf{b}^2}{2\left(1 - \alpha\mathsf{c}\right)},$$

with $\mathsf{a}, \mathsf{b}, \mathsf{c}$ as before. Chernoff entropy is $\eta^* = \max_{0<\alpha<1}\eta(\alpha)$, which is a one-dimensional concave maximization easily done numerically.

(b) If $\mathsf{f}_0 = \mathsf{f}_1 = \mathsf{f}$ then $\mathsf{c} = 0$, $\mathsf{a} = -\mathsf{m}^2/\left(2\mathsf{f}^2\right)$, and $\mathsf{b} = -\mathsf{m}/\mathsf{f}$. The logarithmic term drops and

$$\eta(\alpha) = \frac{\alpha\mathsf{m}^2}{2\mathsf{f}^2} - \frac{\alpha^2\mathsf{m}^2}{2\mathsf{f}^2} = \frac{\alpha(1-\alpha)}{2}\left(\frac{\mathsf{m}}{\mathsf{f}}\right)^2,$$

the same parabola as in Exercise 7.1 with δ replaced by m/f. So $\alpha^* = \frac{1}{2}$ and

$$\eta^* = \frac{\mathsf{m}^2}{8\mathsf{f}^2} = \frac{\left(\mathsf{n}_0 - \mathsf{n}_1\right)^2}{8\mathsf{f}^2}.$$

(c) Raising $\mathsf{f}_0 = \mathsf{f}_1 = \mathsf{f}$ with the means fixed lowers η^* like $1/\mathsf{f}^2$: noisier data make the two models harder to tell apart, and the decay rate of both mistake probabilities falls. Widening $\left|\mathsf{n}_0 - \mathsf{n}_1\right|$ with f fixed raises η^* quadratically: the models make increasingly different predictions and a moderate sample suffices. Only the ratio m/f matters, not its two ingredients separately.

(d) With $\mathsf{n}_0 = \mathsf{n}_1$ we have $\mathsf{m} = 0$, hence $\mathsf{b} = 0$ and $\mathsf{a} = \log\left(\mathsf{f}_0/\mathsf{f}_1\right)$, so

$$\eta(\alpha) = \frac{1}{2}\log\left(1 - \alpha\mathsf{c}\right) - \alpha\log\frac{\mathsf{f}_0}{\mathsf{f}_1}, \qquad \mathsf{c} = 1 - \frac{\mathsf{f}_0^2}{\mathsf{f}_1^2}.$$

This is strictly positive on $(0, 1)$ whenever $\mathsf{f}_0 \neq \mathsf{f}_1$, so models differing only in volatility are still statistically distinguishable, and at an exponential rate. Differentiating, the maximizer solves $\frac{\mathsf{c}}{2(1-\alpha\mathsf{c})} = -\log\frac{\mathsf{f}_0}{\mathsf{f}_1}$, whose solution is not $\frac{1}{2}$ except in the degenerate case $\mathsf{f}_0 = \mathsf{f}_1$.

In the equal-volatility case $\eta(\cdot)$ is symmetric about $\frac{1}{2}$ because the two models are equally far from each other: the roles of numerator and denominator in the likelihood ratio can be exchanged without changing anything. When volatilities differ, that exchange symmetry is broken. A high-volatility model assigns non-negligible probability to data a low-volatility model regards as nearly impossible, but not the reverse, so

> evidence accumulates asymmetrically. Chernoff entropy remains a single number equalizing the two decay rates, as the chapter shows, but it is now attained at an α^* tilted toward one of the models, and that tilt is the quantitative record of the asymmetry.

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[1] This inequality analysis extends pointwise in θ. A pointwise use of the Law of Large Numbers is typically not sufficient to justify statistical consistency.